Formal Concept Analysis: Foundations and Applications

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Outline

Introduction

A formal context

	cartoon	*real*	*tortoise*	*dog*	*cat*	*mammal*
Garfield	$\otimes$				$\otimes$	$\otimes$
Snoopy	$\otimes$			$\otimes$		$\otimes$
Socks		$\otimes$			$\otimes$	$\otimes$
Bobby		$\otimes$		$\otimes$		$\otimes$
Harriet		$\otimes$	$\otimes$			

A formal context

	cartoon	*real*	*tortoise*	*dog*	*cat*	*mammal*
Garfield	$\otimes$				$\otimes$	$\otimes$
Snoopy	$\otimes$			$\otimes$		$\otimes$
Socks		$\otimes$			$\otimes$	$\otimes$
Bobby		$\otimes$		$\otimes$		$\otimes$
Harriet		$\otimes$	$\otimes$			

Introduction

The duality of extension and intension

A formal context

	cartoon	*real*	*tortoise*	*dog*	*cat*	*mammal*
Garfield	$\otimes$				$\otimes$	$\otimes$
Snoopy	$\otimes$			$\otimes$		$\otimes$
Socks		$\otimes$			$\otimes$	$\otimes$
Bobby		$\otimes$		$\otimes$		$\otimes$
Harriet		$\otimes$	$\otimes$			

A formal context

	cartoon	*real*	*tortoise*	*dog*	*cat*	*mammal*
Garfield	$\otimes$				$\otimes$	$\otimes$
Snoopy	$\otimes$			$\otimes$		$\otimes$
Socks		$\otimes$			$\otimes$	$\otimes$
Bobby		$\otimes$		$\otimes$		$\otimes$
Harriet		$\otimes$	$\otimes$			

The pair ({*Garfield*, *Snoopy*}, {*cartoon*, *mammal*}) is a formal concept of the formal context

Introduction

Formal concept analysis in information sciences

- Formal concept analysis in information retrieval
- Formal concept analysis as a tool for knowledge representation and knowledge discovery
- Applications of formal concept analysis in logic and artificial intelligence

Introductions to formal concept analysis

- Davey, B., Priestley, H.: *Introduction to Lattices and Order.* Cambridge University Press (2002, Second Edition)
- Ganter, B., Wille, R.: *Formal Concept Analysis. Mathematical Foundations.* Springer-Verlag (1999)
- www.fcahome.org.uk

International conferences

- International Conference on Conceptual Structures (ICCS)
- International Conference on Formal Concept Analysis (ICFCA)
- Concept Lattices and their Applications (CLA)

Concept lattices of contexts

Concept lattices of contexts

Context and concept

Formal context: structure of the form $\mathbb{K} = (\mathit{Ob}, \mathit{At}, I)$ where

- *Ob* is a nonempty set of formal objects
- *At* is a nonempty set of formal attributes
- *I* is a binary relation between *Ob* and *At*

Concept lattices of contexts

Context and concept

A finite context can be represented by a cross table where

- rows are headed by object names
- columns are headed by attribute names

	x
	$\vdots$
	$\vdots$
X	$\cdots \quad \cdots \quad \otimes \quad \cdots \quad \cdots$
	$\vdots$
	$\vdots$

A cross in row X and column x means that

- the object X has the attribute x

Concept lattices of contexts

Context and concept

Example 1:

	small	*medium*	*large*	*near*	*far*	*yes*	*no*
Mercury	⊗			⊗			⊗
Venus	⊗			⊗			⊗
Earth	⊗			⊗		⊗	
Mars	⊗			⊗		⊗	
Jupiter			⊗		⊗	⊗	
Saturn			⊗		⊗	⊗	
Uranus		⊗			⊗	⊗	
Neptune		⊗			⊗	⊗	
Pluto	⊗				⊗	⊗	

Concept lattices of contexts

Context and concept

Example 1:

	small	*medium*	*large*	*near*	*far*	*yes*	*no*
Mercury	$\otimes$			$\otimes$			$\otimes$
Venus	$\otimes$			$\otimes$			$\otimes$
Earth	$\otimes$			$\otimes$		$\otimes$	
Mars	$\otimes$			$\otimes$		$\otimes$	
Jupiter			$\otimes$		$\otimes$	$\otimes$	
Saturn			$\otimes$		$\otimes$	$\otimes$	
Uranus		$\otimes$			$\otimes$	$\otimes$	
Neptune		$\otimes$			$\otimes$	$\otimes$	
Pluto	$\otimes$				$\otimes$	$\otimes$	

Example 1:

	small	*near*	*medium*	*large*	*far*	*yes*	*no*
Mercury	⊗	⊗					⊗
Venus	⊗	⊗					⊗
Earth	⊗	⊗				⊗	
Mars	⊗	⊗				⊗	
Jupiter				⊗	⊗	⊗	
Saturn				⊗	⊗	⊗	
Uranus			⊗		⊗	⊗	
Neptune			⊗		⊗	⊗	
Pluto	⊗				⊗	⊗	

Concept lattices of contexts
Context and concept

Example 2:

	a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*
1	⊗	⊗					⊗		
2	⊗	⊗					⊗	⊗	
3	⊗	⊗	⊗				⊗	⊗	⊗
4	⊗		⊗				⊗	⊗	
5	⊗	⊗		⊗		⊗			
6	⊗	⊗	⊗	⊗		⊗			
7	⊗		⊗	⊗	⊗				
8	⊗		⊗	⊗		⊗			
9	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗

Example 2:

	a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*
1	⊗	⊗					⊗		
2	⊗	⊗					⊗	⊗	
3	⊗	⊗	⊗				⊗	⊗	⊗
4	⊗		⊗				⊗	⊗	
5	⊗	⊗		⊗		⊗			
6	⊗	⊗	⊗	⊗		⊗			
7	⊗		⊗	⊗	⊗				
8	⊗		⊗	⊗		⊗			
9	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗

Example 2:

	a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*
1	$\otimes$	$\otimes$					$\otimes$		
2	$\otimes$	$\otimes$					$\otimes$	$\otimes$	
3	$\otimes$	$\otimes$	$\otimes$				$\otimes$	$\otimes$	$\otimes$
4	$\otimes$		$\otimes$				$\otimes$	$\otimes$	
5	$\otimes$	$\otimes$		$\otimes$		$\otimes$			
6	$\otimes$	$\otimes$	$\otimes$	$\otimes$		$\otimes$			
7	$\otimes$		$\otimes$	$\otimes$	$\otimes$				
8	$\otimes$		$\otimes$	$\otimes$		$\otimes$			
9	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$

Concept lattices of contexts

Context and concept

Example 2:

	a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*
1	⊗	⊗					⊗		
2	⊗	⊗					⊗	⊗	
3	⊗	⊗	⊗				⊗	⊗	⊗
4	⊗		⊗				⊗	⊗	
5	⊗	⊗		⊗		⊗			
6	⊗	⊗	⊗	⊗		⊗			
7	⊗		⊗	⊗	⊗				
8	⊗		⊗	⊗		⊗			
9	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗

Concept lattices of contexts

Context and concept

Example 2:

	a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*
1	⊗	⊗					⊗		
2	⊗	⊗					⊗	⊗	
3	⊗	⊗	⊗				⊗	⊗	⊗
4	⊗		⊗				⊗	⊗	
5	⊗	⊗		⊗		⊗			
6	⊗	⊗	⊗	⊗		⊗			
7	⊗		⊗	⊗	⊗				
8	⊗		⊗	⊗		⊗			
9	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗

For a set $A \subseteq Ob$ of objects, we define

- $A' = \{x \in At\colon X \; I \; x \text{ for every } X \in A\}$

i.e. the set of attributes common to the objects in A

For a set $B \subseteq At$ of attributes, we define

- $B' = \{X \in Ob\colon X \; I \; x \text{ for every } x \in B\}$

i.e. the set of objects which have all attributes in B

Proposition 1: If (Ob, At, I) is a context, $A, A_1, A_2 \subseteq Ob$ are sets of objects and $B, B_1, B_2 \subseteq At$ are sets of attributes then

- $A_1 \subseteq A_2 \Rightarrow A_2' \subseteq A_1'$
- $B_1 \subseteq B_2 \Rightarrow B_2' \subseteq B_1'$
- $A \subseteq A''$
- $B \subseteq B''$
- $A' = A'''$
- $B' = B'''$

Moreover,

- $A \subseteq B' \Leftrightarrow B \subseteq A' \Leftrightarrow A \times B \subseteq I$

Concept lattices of contexts

Context and concept

A formal concept of the context (Ob, At, I) is a pair (A, B) with

- $A \subseteq Ob$
- $B \subseteq At$
- $A' = B$
- $B' = A$

We call

- A the extent of the concept (A, B)
- B the intent of the concept (A, B)

$\mathcal{B}(Ob, At, I)$ denotes

- the set of all concepts of the context (Ob, At, I)

Example 2:

	a	b	c	d	e	f	g	h	i
1	⊗	⊗					⊗		
2	⊗	⊗					⊗	⊗	
3	⊗	⊗	⊗				⊗	⊗	⊗
4	⊗		⊗				⊗	⊗	
5	⊗	⊗		⊗		⊗			
6	⊗	⊗	⊗	⊗		⊗			
7	⊗		⊗	⊗	⊗				
8	⊗		⊗	⊗		⊗			
9	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗

Concept lattices of contexts

Context and concept

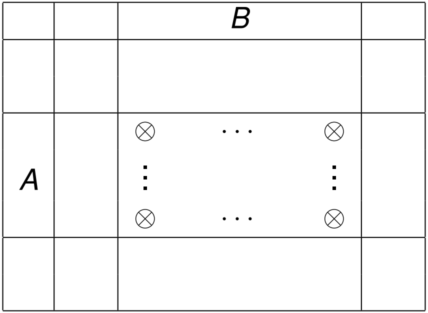
Example 2:

	a	*b*	*g*	*c*	*d*	*e*	*f*	*h*	*i*
1	⊗	⊗	⊗						
2	⊗	⊗	⊗					⊗	
3	⊗	⊗	⊗	⊗				⊗	⊗
4	⊗		⊗	⊗				⊗	
5	⊗	⊗			⊗		⊗		
6	⊗	⊗		⊗	⊗		⊗		
7	⊗			⊗	⊗	⊗			
8	⊗			⊗	⊗		⊗		
9	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗

Example 2:

	a	*b*	*g*	*c*	*d*	*e*	*f*	*h*	*i*
1	⊗	⊗	⊗						
2	⊗	⊗	⊗					⊗	
3	⊗	⊗	⊗	⊗				⊗	⊗
9	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗
4	⊗		⊗	⊗				⊗	
5	⊗	⊗			⊗		⊗		
6	⊗	⊗		⊗	⊗		⊗		
7	⊗			⊗	⊗	⊗			
8	⊗			⊗	⊗		⊗		

The extent A and the intent B of a concept (A, B) are closely connected by the relation I



Concept lattices of contexts

Context and concept

For every set $A \subseteq Ob$,

- A' is an intent of some concept
- (A'', A') is a concept
- A'' is the smallest extent containing A
- A is an extent iff $A = A''$

For every set $B \subseteq At$,

- B' is an extent of some concept
- (B', B'') is a concept
- B'' is the smallest intent containing B
- B is an intent iff $B = B''$

Proposition 2: If T is an index set and for every $t \in T$, $A_t \subseteq Ob$ is a set of objects and $B_t \subseteq At$ is a set of attributes then

- $(\bigcup_{t \in T} A_t)' = \bigcap_{t \in T} A_t'$
- $(\bigcup_{t \in T} B_t)' = \bigcap_{t \in T} B_t'$

Concept lattices of contexts

Context and concept

If (A_1, B_1) and (A_2, B_2) are concepts of a context then

- $A_1 \subseteq A_2$ iff $B_2 \subseteq B_1$

If $A_1 \subseteq A_2$ and $B_2 \subseteq B_1$ then we say that

- (A_1, B_1) is a subconcept of (A_2, B_2)
- (A_2, B_2) is a superconcept of (A_1, B_1)

and we write

- $(A_1, B_1) \leqslant (A_2, B_2)$

The set of all concepts of (Ob, At, I) ordered in this way

- is denoted by $\underline{\mathcal{B}}(Ob, At, I)$
- is called the concept lattice of the context (Ob, At, I)

Concept lattices of contexts

Context and concept

Theorem 1: The concept lattice $\underline{\mathcal{B}}(Ob, At, I)$ is a complete lattice in which infimum and supremum are given by

- $\bigwedge_{t \in T}(A_t, B_t) = (\bigcap_{t \in T} A_t, (\bigcup_{t \in T} B_t)'')$
- $\bigvee_{t \in T}(A_t, B_t) = ((\bigcup_{t \in T} A_t)'', \bigcap_{t \in T} B_t)$

Theorem 2: Every complete lattice $(L, \leqslant)$ is isomorphic to the concept lattice $\underline{\mathcal{B}}(L, L, \leqslant)$

The duality principle for concept lattices: If (Ob, At, I) is a context then

- (At, Ob, I^{-1}) is a context

Moreover,

- $\underline{\mathcal{B}}(At, Ob, I^{-1})$ and $\underline{\mathcal{B}}(Ob, At, I)$ are isomorphic
- $(B, A) \mapsto (A, B)$ is an isomorphism

For an object $X \in Ob$, we write

- X' instead of the object intent $\{X\}'$
- γX for the object concept (X'', X')

For an attribute $x \in At$, we write

- x' instead of the attribute extent $\{x\}'$
- μx for the attribute concept (x', x'')

Concept lattices of contexts

Context and concept lattice

A context can be reconstructed from its concept lattice:

- *Ob* is the extent of the greatest concept $(\emptyset', \emptyset'')$
- *At* is the intent of the least concept $(\emptyset'', \emptyset')$
- *I* is given by
  - $I = \bigcup\{A \times B\colon (A, B) \text{ is a concept}\}$

The contexts reconstructed from two non-isomorphic concept lattices are non-isomorphic

Example 3:

- Concept lattices of non-isomorphic contexts can well be isomorphic

	a	b	c	d	e
1	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$
2	$\otimes$	$\otimes$			
3	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$
4		$\otimes$	$\otimes$	$\otimes$	$\otimes$
5		$\otimes$			
6	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$
7	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$
8		$\otimes$	$\otimes$	$\otimes$	

	a	b	c, d	e
1, 3, 6, 7	$\otimes$	$\otimes$	$\otimes$	$\otimes$
2	$\otimes$	$\otimes$		
4		$\otimes$	$\otimes$	$\otimes$
5		$\otimes$		
8		$\otimes$	$\otimes$	

Concept lattices of contexts

Context and concept lattice

A context (Ob, At, I) is called clarified iff for every object $X, Y \in Ob$ and for every attribute $x, y \in At$,

- $X' = Y' \Rightarrow X = Y$
- $x' = y' \Rightarrow x = y$

If $X \in Ob$ is an object and $A \subseteq Ob$ is a set of objects with $X \notin A$ but $X' = A'$ then

- $\gamma X = \bigvee_{Y \in A} \gamma Y$
- $\underline{\mathcal{B}}(Ob, At, I)$ and $\underline{\mathcal{B}}(Ob \setminus \{X\}, At, I \cap ((Ob \setminus \{X\}) \times At))$ are isomorphic

and we say that

- X is a reducible object

Full rows, i.e.

- objects X with $X' = At$

are always reducible

Concept lattices of contexts

Context and concept lattice

If $x \in At$ is an attribute and $B \subseteq At$ is a set of attributes with $x \notin B$ but $x' = B'$ then

- $\mu x = \bigwedge_{y \in B} \mu y$
- $\underline{\mathcal{B}}(Ob, At, I)$ and $\underline{\mathcal{B}}(Ob, At \setminus \{x\}, I \cap (Ob \times (At \setminus \{x\})))$ are isomorphic

and we say that

- x is a reducible attribute

Full columns, i.e.

- attributes x with $x' = Ob$

are always reducible

Concept lattices of contexts

Context and concept lattice

Example 4:

	a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*
1	⊗	⊗					⊗		
2	⊗	⊗					⊗	⊗	
3	⊗	⊗	⊗				⊗	⊗	⊗
4	⊗		⊗				⊗	⊗	
5	⊗	⊗		⊗		⊗			
6	⊗	⊗	⊗	⊗		⊗			
7	⊗		⊗	⊗	⊗				
8	⊗		⊗	⊗		⊗			
9	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗

Concept lattices of contexts

Context and concept lattice

Example 4:

	a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*
1	⊗	⊗					⊗		
2	⊗	⊗					⊗	⊗	
3	⊗	⊗	⊗				⊗	⊗	⊗
4	⊗		⊗				⊗	⊗	
5	⊗	⊗		⊗		⊗			
6	⊗	⊗	⊗	⊗		⊗			
7	⊗		⊗	⊗	⊗				
8	⊗		⊗	⊗		⊗			
9	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗

The removal from context (Ob, At, I) of reducible objects and reducible attributes is called

- reducing the context

A clarified context (Ob, At, I)

- is called row reduced iff every object concept is irreducible
- is called column reduced iff every attribute concept is irreducible

A clarified context which is both row reduced and column reduced

- is called reduced

Every finite context can be brought into a reduced form

- merge objects with the same intents
- merge attributes with the same extents
- delete all reducible objects
- delete all reducible attributes

Concept lattices of contexts

Context and concept lattice

If (Ob, At, I) is a context, $X \in Ob$ is an object and $x \in At$ is an attribute then we write

- $X \swarrow x$ iff
  - not $X \; I \; x$
  - for every object $Y \in Ob$, if $X' \subsetneq Y'$ then $Y \; I \; x$

In other words,

- $X \swarrow x$ iff
  - X' is maximal among all object intents not containing x

If (Ob, At, I) is a context, $X \in Ob$ is an object and $x \in At$ is an attribute then we write

- $X \nearrow x$ iff
  - not $X \; I \; x$
  - for every attribute $y \in At$, if $x' \subsetneq y'$ then $X \; I \; y$

In other words,

- $X \nearrow x$ iff
  - x' is maximal among all attribute extents not containing X

Concept lattices of contexts

Context and concept lattice

Proposition 3: The following statements hold for every context:

- $X \in Ob$ is irreducible $\Leftrightarrow X \swarrow x$ for some $x \in At$
- $x \in At$ is irreducible $\Leftrightarrow X \nearrow x$ for some $X \in Ob$

Proposition 4: The following statements hold for every finite context:

- $X \in Ob$ is irreducible $\Leftrightarrow X \swarrow\!\!\nearrow x$ for some $x \in At$
- $x \in At$ is irreducible $\Leftrightarrow X \swarrow\!\!\nearrow x$ for some $X \in Ob$

Example 5:

	a	b	c, d	e
$1, 3, 6, 7$	$\otimes$	$\otimes$	$\otimes$	$\otimes$
2	$\otimes$	$\otimes$		
4		$\otimes$	$\otimes$	$\otimes$
5		$\otimes$		
8		$\otimes$	$\otimes$	

Example 5:

	a	b	c, d	e
$1, 3, 6, 7$	$\otimes$	$\otimes$	$\otimes$	$\otimes$
2	$\otimes$	$\otimes$	$\swarrow\ \nearrow$	$\swarrow$
4	$\swarrow\ \nearrow$	$\otimes$	$\otimes$	$\otimes$
5	$\nearrow$	$\otimes$	$\nearrow$	
8	$\nearrow$	$\otimes$	$\otimes$	$\swarrow\ \nearrow$

Example 5:

	a	c, d	e
2	$\otimes$	$\swarrow \nearrow$	$\swarrow$
4	$\swarrow \nearrow$	$\otimes$	$\otimes$
8	$\nearrow$	$\otimes$	$\swarrow \nearrow$

Concept lattices of contexts

Context and concept lattice

A context (Ob, At, I) is called doubly founded iff for every object $X \in Ob$ and for every attribute $x \in At$, if not $X \; I \; x$ then

- $X \nearrow y$ and $x' \subseteq y'$ for some attribute $y \in At$
- $Y \swarrow x$ and $X' \subseteq Y'$ for some object $Y \in Ob$

Proposition 5: Every finite context is doubly founded

Proposition 6: A context which does neither contain infinite chains $X_1, X_2, \ldots$ of objects with $X_1' \subseteq X_2' \subseteq \ldots$ nor infinite chains $x_1, x_2, \ldots$ of attributes with $x_1' \subseteq x_2' \subseteq \ldots$ is doubly founded

Proposition 7: The following statements hold for every doubly founded context:

- $X \swarrow x \Rightarrow X \swarrow\!\!\nearrow y$ for some $y \in At$
- $X \nearrow x \Rightarrow Y \swarrow\!\!\nearrow x$ for some $Y \in Ob$

A complete lattice $(L, \leqslant)$ is called doubly founded iff for every $u, v \in L$, if $u < v$ then there exists $u', v' \in L$ such that

- u' is minimal with respect to $u' \nleqslant u$ and $u' \leqslant v$
- v' is maximal with respect to $u \leqslant v'$ and $v \nleqslant v'$

Proposition 8: If the concept lattice of the context (Ob, At, I) is doubly founded, so is (Ob, At, I)

Proposition 9: If the complete lattice $(L, \leqslant)$ is not doubly founded, neither is the context $(L, L, \leqslant)$

Many-valued contexts

Many-valued contexts

Many-valued context: structure of the form (Ob, At, Va, I) where

- Ob is a nonempty set of formal objects
- At is a nonempty set of formal attributes
- Va is a nonempty set of formal values
- I is a ternary relation between Ob, At and Va

A many-valued context (Ob, At, Va, I)

- is called a n-valued context iff Va has n elements

Example 6:

	De	*Dl*	*R*	*E*	*M*
Conv.	*poor*	*good*	*good*	*good*	*excellent*
Front	*good*	*poor*	*excellent*	*excellent*	*good*
Rear	*excellent*	*excellent*	*very poor*	*poor*	*very poor*
Mid	*excellent*	*excellent*	*good*	*very poor*	*very poor*
All	*excellent*	*excellent*	*good*	*good*	*poor*

De: “drive efficiency empty”, *Dl*: “drive efficiency loaded”, *R*: “road handling properties”, *E*: “economy of space”, *M*: “maintainability”

The domain of an attribute x is defined to be

- $dom(x) = \{X \in Ob\colon I(X, x, v) \text{ for some } v \in Va\}$

The attribute x

- is called complete iff $dom(x) = Ob$

A many-valued context

- is called complete iff all its attributes are complete

A scale for the attribute x of a many-valued context

- is a (one-valued) context $\mathcal{K}_x = (Ob_x, At_x, I_x)$ with $\{v \in Va\colon I(X, x, v) \text{ for some } X \in Ob\} \subseteq Ob_x$

If (Ob, At, Va, I) is a many-valued context and (Ob_x, At_x, I_x) is a scale context for every $x \in At$ then

- the derived context with respect to plain scaling is the (one-valued) context (Ob', At', I') with
  - $Ob' = Ob$
  - $At' = \{(x, a)\colon x \in At \text{ and } a \in At_x\}$
  - $X\ I'\ (x, a)$ iff $I(X, x, v)$ and $v\ I_x\ a$ for some $v \in Va$

Example 7:

	De	*Dl*	*R*	*E*	*M*
Conv.	*poor*	*good*	*good*	*good*	*excellent*
Front	*good*	*poor*	*excellent*	*excellent*	*good*
Rear	*excellent*	*excellent*	*very poor*	*poor*	*very poor*
Mid	*excellent*	*excellent*	*good*	*very poor*	*very poor*
All	*excellent*	*excellent*	*good*	*good*	*poor*

$\mathcal{K}_{De}, \mathcal{K}_{Dl}, \mathcal{K}_{R}, \mathcal{K}_{E}, \mathcal{K}_{M}$:

	$++$	$+$	$-$	$--$
excellent	$\otimes$	$\otimes$		
good		$\otimes$		
poor			$\otimes$	
very poor			$\otimes$	$\otimes$

If $\mathcal{K} = (Ob, At, I)$ is a context then we define

- $\mathcal{K}^c = (Ob, At, (Ob \times At) \setminus I)$
- $\mathcal{K}^{-1} = (At, Ob, I^{-1})$

If $\mathcal{K}_1 = (Ob_1, At_1, I_1)$ and $\mathcal{K}_2 = (Ob_2, At_2, I_2)$ are contexts then we define for every $i \in \{1, 2\}$,

- $\dot{O}b_i = \{i\} \times Ob_i$
- $\dot{A}t_i = \{i\} \times At_i$
- $(i, X)\ \dot{I}_i\ (i, x)$ iff $X\ I_i\ x$

If $\mathcal{K}_1 = (Ob_1, At_1, I_1)$ and $\mathcal{K}_2 = (Ob_2, At_2, I_2)$ are contexts then we define

- $\mathcal{K}_1 \dot{\cup}_I \mathcal{K}_2 = (\dot{O}b_1 \cup \dot{O}b_2, At_1 \cup At_2, I)$

with

- $(i, X)\ I\ x$ iff $x \in At_i$ and $X\ I_i\ x$

Example 17:

	1	2
a	⊗	
b		⊗
c		⊗

$\dot{\cup}_I$

	1	2
d	⊗	⊗
e	⊗	

$=$

	1	2
a	⊗	
b		⊗
c		⊗
d	⊗	⊗
e	⊗	

Many-valued contexts

Context constructions and standard scales

If $\mathcal{K}_1 = (Ob_1, At_1, I_1)$ and $\mathcal{K}_2 = (Ob_2, At_2, I_2)$ are contexts then we define

- $\mathcal{K}_1 \dot{\cup}_r \mathcal{K}_2 = (Ob_1 \cup Ob_2, \dot{A}t_1 \cup \dot{A}t_2, I)$

with

- $X\ I\ (i, x)$ iff $X \in Ob_i$ and $X\ I_i\ x$

Many-valued contexts

Context constructions and standard scales

Example 18:

	1	2
a	$\otimes$	
b		$\otimes$

$\dot\cup_r$

	3	4	5
a	$\otimes$	$\otimes$	
b		$\otimes$	$\otimes$

$=$

	1	2	3	4	5
a	$\otimes$		$\otimes$	$\otimes$	
b		$\otimes$		$\otimes$	$\otimes$

If $\mathcal{K}_1 = (Ob_1, At_1, I_1)$ and $\mathcal{K}_2 = (Ob_2, At_2, I_2)$ are contexts then we define

- $\mathcal{K}_1 \dot{\cup} \mathcal{K}_2 = (\dot{O}b_1 \cup \dot{O}b_2, \dot{A}t_1 \cup \dot{A}t_2, \dot{I}_1 \cup \dot{I}_2)$

Example 19:

	1	2
a	$\otimes$	
b		$\otimes$
c		$\otimes$

$\dot\cup$

	3	4	5
d	$\otimes$	$\otimes$	
e		$\otimes$	$\otimes$

$=$

	1	2	3	4	5
a	$\otimes$				
b		$\otimes$			
c		$\otimes$			
d			$\otimes$	$\otimes$	
e				$\otimes$	$\otimes$

Nominal scales: $\mathbb{N}_k = (\{1, \ldots, k\}, \{1, \ldots, k\}, =)$

Example 20:

$\mathbb{N}_4$:

	1	2	3	4
1	$\otimes$			
2		$\otimes$		
3			$\otimes$	
4				$\otimes$

Ordinal scales: $\mathbb{O}_k = (\{1,\ldots,k\},\{1,\ldots,k\},\leqslant)$

Example 21:

$\mathbb{O}_4$:

	1	2	3	4
1	$\otimes$	$\otimes$	$\otimes$	$\otimes$
2		$\otimes$	$\otimes$	$\otimes$
3			$\otimes$	$\otimes$
4				$\otimes$

Interordinal scales:

$\mathbb{I}_k = (\{1,\ldots,k\},\{1,\ldots,k\},\leqslant)\dot{\cup}_r(\{1,\ldots,k\},\{1,\ldots,k\},\geqslant)$

Example 22:

$\mathbb{I}_4$:

	$\leqslant 1$	$\leqslant 2$	$\leqslant 3$	$\leqslant 4$	$\geqslant 1$	$\geqslant 2$	$\geqslant 3$	$\geqslant 4$
1	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$			
2		$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$		
3			$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$	
4				$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$

Biordinal scales:

$\mathbb{M}_{k,l} = (\{1, \ldots, k\}, \{1, \ldots, k\}, \leqslant) \dot{\cup} (\{1, \ldots, l\}, \{1, \ldots, l\}, \geqslant)$

Example 23:

$\mathbb{M}_{4,2}$:

	$\leqslant 1$	$\leqslant 2$	$\leqslant 3$	$\leqslant 4$	$\geqslant 5$	$\geqslant 6$
1	$\otimes$	$\otimes$	$\otimes$	$\otimes$		
2		$\otimes$	$\otimes$	$\otimes$		
3			$\otimes$	$\otimes$		
4				$\otimes$		
5					$\otimes$	
6					$\otimes$	$\otimes$

Many-valued contexts

Context constructions and standard scales

Dichotomic scale: $\mathbb{D} = (\{0,1\},\{0,1\},=)$

$\mathbb{D}$:

	0	1
0	$\otimes$	
1		$\otimes$

If $\mathcal{K}_1 = (Ob_1, At_1, I_1)$ and $\mathcal{K}_2 = (Ob_2, At_2, I_2)$ are contexts then we define

- $\mathcal{K}_1 + \mathcal{K}_2 = (\dot{O}b_1 \cup \dot{O}b_2, \dot{A}t_1 \cup \dot{A}t_2, I)$

with

- $(i, X)\ I\ (j, x)$ iff one of the following conditions hold
  - $i = 1$, $j = 1$ and $X\ I_1\ x$
  - $i = 1$ and $j = 2$
  - $i = 2$ and $j = 1$
  - $i = 2$, $j = 2$ and $X\ I_2\ x$

Example 24:

	1	2
a	⊗	
b		⊗
c		⊗

+

	3	4	5
d		⊗	
e	⊗		⊗

=

	1	2	3	4	5
a	⊗		⊗	⊗	⊗
b		⊗	⊗	⊗	⊗
c		⊗	⊗	⊗	⊗
d	⊗	⊗		⊗	
e	⊗	⊗	⊗		⊗

Proposition 17: If $\mathcal{K}_1 = (Ob_1, At_1, I_1)$ and $\mathcal{K}_2 = (Ob_2, At_2, I_2)$ are contexts then

- $\underline{\mathcal{B}}(\mathcal{K}_1 + \mathcal{K}_2)$ and $\underline{\mathcal{B}}(\mathcal{K}_1) \times \underline{\mathcal{B}}(\mathcal{K}_2)$ are isomorphic
- $(A, B) \mapsto ((A \cap \dot{O}b_1, B \cap \dot{A}t_1), (A \cap \dot{O}b_2, B \cap \dot{A}t_2))$ is an isomorphism

Many-valued contexts

Context constructions and standard scales

If $\mathcal{K}_1 = (Ob_1, At_1, I_1)$ and $\mathcal{K}_2 = (Ob_2, At_2, I_2)$ are contexts then we define

- $\mathcal{K}_1 \bowtie \mathcal{K}_2 = (Ob_1 \times Ob_2, \dot{A}t_1 \cup \dot{A}t_2, I)$

with

- $(X_1, X_2)\ I\ (i, x)$ iff $X_i\ I_i\ x$

Example 25:

	1	2
a	$\otimes$	
b		$\otimes$
c		$\otimes$

$\bowtie$

	3	4	5
d		$\otimes$	
e	$\otimes$		$\otimes$

$=$

	1	2	3	4	5
(a, d)	$\otimes$			$\otimes$	
(a, e)	$\otimes$		$\otimes$		$\otimes$
(b, d)		$\otimes$		$\otimes$	
(b, e)		$\otimes$	$\otimes$		$\otimes$
(c, d)		$\otimes$		$\otimes$	
(c, e)		$\otimes$	$\otimes$		$\otimes$

Proposition 18: If $\mathcal{K}_1 = (Ob_1, At_1, I_1)$ and $\mathcal{K}_2 = (Ob_2, At_2, I_2)$ are contexts then

- the extents of $\mathcal{K}_1 \bowtie \mathcal{K}_2$ are precisely the sets of the form $A_1 \times A_2$ each set A_i being an extent of $\mathcal{K}_i$

Many-valued contexts

Context constructions and standard scales

If $\mathcal{K}_1 = (Ob_1, At_1, I_1)$ and $\mathcal{K}_2 = (Ob_2, At_2, I_2)$ are contexts then we define

- $\mathcal{K}_1 \times \mathcal{K}_2 = (Ob_1 \times Ob_2, At_1 \times At_2, I)$

with

- $(X_1, X_2)\ I\ (x_1, x_2)$ iff $X_1\ I_1\ x_1$ or $X_2\ I_2\ x_2$

Example 26:

	1	2
a	$\otimes$	
b		$\otimes$

$\times$

	3	4
c	$\otimes$	
d	$\otimes$	$\otimes$

$=$

	$(1,3)$	$(1,4)$	$(2,3)$	$(2,4)$
(a,c)	$\otimes$	$\otimes$	$\otimes$	
(a,d)	$\otimes$	$\otimes$	$\otimes$	$\otimes$
(b,c)	$\otimes$		$\otimes$	$\otimes$
(b,d)	$\otimes$	$\otimes$	$\otimes$	$\otimes$

Proposition 19: If $\mathcal{K}_1 = (Ob_1, At_1, I_1)$, $\mathcal{K}_2 = (Ob_2, At_2, I_2)$ and $\mathcal{K}_3 = (Ob_3, At_3, I_3)$ are contexts then

- $(\mathcal{K}_1 + \mathcal{K}_2) \times \mathcal{K}_3$ and $(\mathcal{K}_1 \times \mathcal{K}_3) + (\mathcal{K}_2 \times \mathcal{K}_3)$ are isomorphic

Contranominal scales: $\mathbb{N}^c_S = (S, S, \neq)$ for every nonempty set S

Example 27:

$\mathbb{N}^c_{\{1,2,3\}}$:

	1	2	3
1		⊗	⊗
2	⊗		⊗
3	⊗	⊗	

Proposition 20: If S is a nonempty set then

- the concepts of $\mathbb{N}^c_S$ are precisely the pairs $(A, S \setminus A)$ for $A \subseteq S$

General ordinal scales: $\mathbb{O}_P = (P, P, \leqslant)$ for every ordered set $(P, \leqslant)$

Example 28:

$\mathbb{O}_{\{1,2,3\}}$:

	1	2	3
1	$\otimes$	$\otimes$	$\otimes$
2		$\otimes$	$\otimes$
3			$\otimes$

Proposition 21: If $(P, \leqslant)$ is an ordered set then

- the concepts of $\mathbb{O}_P$ are precisely the pairs (A, B) where A is the set of all lower bounds of B and B is the set of all upper bounds of A

Contraordinal scales: $\mathbb{O}_P^{cd} = (P, P, \ngeq)$ for every ordered set $(P, \leqslant)$

Example 29:

$\mathbb{O}^{cd}_{\{1,2,3\}}$:

	1	2	3
1		$\otimes$	$\otimes$
2			$\otimes$
3			

Proposition 22: If $(P, \leqslant)$ is an ordered set then

- the concepts of $\mathbb{O}_P^{cd}$ are precisely the pairs $(A, P \setminus A)$ for $A \subseteq P$ an order ideal

Contraordinal scales: $\mathbb{O}_S = (2^S, 2^S, \not\supseteq)$ for every set S

Example 30:

$\mathbb{O}_{\{a,b\}}$:

	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$
$\emptyset$		$\otimes$	$\otimes$	$\otimes$
$\{a\}$			$\otimes$	$\otimes$
$\{b\}$		$\otimes$		$\otimes$
$\{a,b\}$				

From an ordered set $(P, \leqslant)$, we obtain the general interordinal scale

- $\mathbb{I}_P = (P, P, \leqslant) \cup_r (P, P, \geqslant)$

and the convex-ordinal scale

- $\mathbb{I}_P = (P, P, \not\geqslant) \cup_r (P, P, \not\leqslant)$

Many-valued contexts
Indiscernibility

If (Ob, At, Va, I) is a complete many-valued context, with every susbset of attributes $B \subseteq At$, we associate a binary relation $IND(B)$, called an indiscernibility relation and defined thus

- $IND(B) = \{(X, Y) \in Ob \times Ob$: for every $x \in B$ and for every $v \in Va$, $I(X, x, v)$ iff $I(Y, x, v)\}$

For an attribute $x \in Att$, we write

- $IND(x)$ instead of $IND(\{x\})$

Obviously $IND(B)$ is an equivalence relation and

- $IND(B) = \bigcap\{IND(x)\colon x \in B\}$

Many-valued contexts

Indiscernibility

Example 8:

	a	*b*	*c*	*d*	*e*
1	1	0	2	2	0
2	0	1	1	1	2
3	2	0	0	1	1
4	1	1	0	2	2
5	1	0	2	0	1
6	2	2	0	1	1
7	2	1	1	1	2
8	0	1	1	0	1

Exemplary partitions generated by attributes in this context

- $Ob/IND(a) = \{\{1,4,5\},\{2,8\},\{3,6,7\}\}$
- $Ob/IND(b) = \{\{1,3,5\},\{2,4,7,8\},\{6\}\}$

Many-valued contexts

Indiscernibility

Approximations of sets of objects in a complete many-valued context (Ob, At, Va, I): with each subset of objects $A \subseteq Ob$ and each subset of attributes $B \subseteq At$, we associate two subsets

- $\underline{IND(B)}(A) = \{X \in Ob\colon IND(B)(X) \subseteq A\}$
- $\overline{IND(B)}(A) = \{X \in Ob\colon IND(B)(X) \cap A \neq \emptyset\}$

called the $IND(B)$-lower approximation of A and the $IND(B)$-upper approximation of A

Obviously

- $\underline{IND(B)}(A) \subseteq A \subseteq \overline{IND(B)}(A)$

Given a subset of objects $A \subseteq Ob$ and a subset of attributes $B \subseteq At$, we shall say that

- A is $IND(B)$-definable iff $\underline{IND(B)}(A) = \overline{IND(B)}(A)$
- A is $IND(B)$-rough iff $\underline{IND(B)}(A) \neq \overline{IND(B)}(A)$

Given a subset of objects $A \subseteq Ob$ and a subset of attributes $B \subseteq At$, let us observe that

- $\underline{IND(B)}(A)$ is the maximal $IND(B)$-definable set of objects contained in A
- $\overline{IND(B)}(A)$ is the minimal $IND(B)$-definable set of objects containing A

Many-valued contexts

Indiscernibility

Example 9:

	a	b	c	d	e
1	1	0	2	2	0
2	0	1	1	1	2
3	2	0	0	1	1
4	1	1	0	2	2
5	1	0	2	0	1
6	2	2	0	1	1
7	2	1	1	1	2
8	0	1	1	0	1

If $A = \{1, 2, 3, 4, 5\}$ and $B = \{a, b, c\}$ then

- $\underline{IND(B)}(A) = \{1, 3, 4, 5\}$
- $\overline{IND(B)}(A) = \{1, 2, 3, 4, 5, 8\}$

Many-valued contexts

Indiscernibility

Proposition 10:

1. $\underline{IND(B)}(\emptyset) = \emptyset$
2. $\underline{IND(B)}(Ob) = Ob$
3. $\underline{IND(B)}(A_1 \cup A_2) \supseteq \underline{IND(B)}(A_1) \cup \underline{IND(B)}(A_2)$
4. $\underline{IND(B)}(A_1 \cap A_2) = \underline{IND(B)}(A_1) \cap \underline{IND(B)}(A_2)$
5. $\underline{IND(B)}(Ob \setminus A) = Ob \setminus \overline{IND(B)}(A)$
6. $A_1 \subseteq A_2$ implies $\underline{IND(B)}(A_1) \subseteq \underline{IND(B)}(A_2)$
7. $\underline{IND(B)}(\underline{IND(B)}(A)) = \underline{IND(B)}(A)$
8. $\underline{IND(B)}(\overline{IND(B)}(A)) = \overline{IND(B)}(A)$

Many-valued contexts

Indiscernibility

Example 10:

- Suppose we are given a complete many-valued context (Ob, At, Va, I) where $Ob = \{1,2,3,4,5,6,7,8\}$ and let $B \subseteq At$ be a subset of attributes defining an equivalence relation $IND(B)$ with the following equivalence classes:
  - $\{1,4,8\}$
  - $\{2,5,7\}$
  - $\{3\}$
  - $\{6\}$
- If $A_1 = \{1,4,7\}$ and $A_2 = \{2,8\}$ then
  - $\underline{IND(B)}(A_1) = \emptyset$
  - $\underline{IND(B)}(A_2) = \emptyset$
  - $\underline{IND(B)}(A_1 \cup A_2) = \{1,4,8\}$

Many-valued contexts

Indiscernibility

Proposition 11:

1. $\overline{IND(B)}(\emptyset) = \emptyset$
2. $\overline{IND(B)}(Ob) = Ob$
3. $\overline{IND(B)}(A_1 \cup A_2) = \overline{IND(B)}(A_1) \cup \overline{IND(B)}(A_2)$
4. $\overline{IND(B)}(A_1 \cap A_2) \subseteq \overline{IND(B)}(A_1) \cap \overline{IND(B)}(A_2)$
5. $\overline{IND(B)}(Ob \setminus A) = Ob \setminus \underline{IND(B)}(A)$
6. $A_1 \subseteq A_2$ implies $\overline{IND(B)}(A_1) \subseteq \overline{IND(B)}(A_2)$
7. $\overline{IND(B)}(\underline{IND(B)}(A)) = \underline{IND(B)}(A)$
8. $\overline{IND(B)}(\overline{IND(B)}(A)) = \overline{IND(B)}(A)$

Example 11:

- Suppose we are given a complete many-valued context (Ob, At, Va, I) where $Ob = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and let $B \subseteq At$ be a subset of attributes defining an equivalence relation $IND(B)$ with the following equivalence classes:
  - $\{1, 4, 8\}$
  - $\{2, 5, 7\}$
  - $\{3\}$
  - $\{6\}$
- If $A_1 = \{1, 3, 5\}$ and $A_2 = \{2, 3, 4, 6\}$ then
  - $\overline{IND(B)}(A_1) = \{1, 2, 3, 4, 5, 7, 8\}$
  - $\overline{IND(B)}(A_2) = \{1, 2, 3, 4, 5, 6, 7, 8\}$
  - $\overline{IND(B)}(A_1 \cap A_2) = \{3\}$

Many-valued contexts

Indiscernibility

Given a complete many-valued context (Ob, At, Va, I), a subset of objects $A \subseteq Ob$ and a subset of attributes $B \subseteq At$, let

- $X \; \underline{in(B)} \; A$ iff $X \in \underline{IND(B)}(A)$
- $X \; \overline{in(B)} \; A$ iff $X \in \overline{IND(B)}(A)$

Intuitive reading

- $X \; \underline{in(B)} \; A$: "$X$ surely belongs to A with respect to B"
- $X \; \overline{in(B)} \; A$: "$X$ possibly belongs to A with respect to B"

Obviously

- $X \; \underline{in(B)} \; A$ implies $X \in A$
- $X \in A$ implies $X \; \overline{in(B)} \; A$

Proposition 12:

1. not $X\ \underline{in(B)}\ \emptyset$
2. $X\ \underline{in(B)}\ Ob$
3. $X\ \underline{in(B)}\ (A_1 \cup A_2)$ if $X\ \underline{in(B)}\ A_1$ or $X\ \underline{in(B)}\ A_2$
4. $X\ \underline{in(B)}\ (A_1 \cap A_2)$ iff $X\ \underline{in(B)}\ A_1$ and $X\ \underline{in(B)}\ A_2$
5. $X\ \underline{in(B)}\ (Ob \setminus A)$ iff not $X\ \overline{in(B)}\ A$
6. $A_1 \subseteq A_2$ implies $X\ \underline{in(B)}\ A_1$ only if $X\ \underline{in(B)}\ A_2$

Proposition 13:

1. not $X\ \overline{in(B)}\ \emptyset$
2. $X\ \overline{in(B)}\ Ob$
3. $X\ \overline{in(B)}\ (A_1 \cup A_2)$ iff $X\ \overline{in(B)}\ A_1$ or $X\ \overline{in(B)}\ A_2$
4. $X\ \overline{in(B)}\ (A_1 \cap A_2)$ only if $X\ \overline{in(B)}\ A_1$ and $X\ \overline{in(B)}\ A_2$
5. $X\ \overline{in(B)}\ (Ob \setminus A)$ iff not $X\ \underline{in(B)}\ A$
6. $A_1 \subseteq A_2$ implies $X\ \overline{in(B)}\ A_1$ only if $X\ \overline{in(B)}\ A_2$

Given a complete many-valued context (Ob, At, Va, I), a subset of objects $A \subseteq Ob$ and a subset of attributes $B \subseteq At$, let

- $BN(B)(A) = \overline{IND(B)}(A) \setminus \underline{IND(B)}(A)$

be the $IND(B)$-boundary of A

Obviously

- A is $IND(B)$-definable iff $BN(B)(A) = \emptyset$
- A is $IND(B)$-rough iff $BN(B)(A) \neq \emptyset$

Many-valued contexts

Indiscernibility

Example 12:

- Suppose we are given a complete many-valued context (Ob, At, Va, I) where $Ob = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and let $B \subseteq At$ be a subset of attributes defining an equivalence relation $IND(B)$ with the following equivalence classes:
  - $\{1, 4, 8\}$
  - $\{2, 5, 7\}$
  - $\{3\}$
  - $\{6\}$
- If $A_1 = \{1, 4, 7\}$ and $A_2 = \{2, 8\}$ then
  - $BN(B)(A_1) = \{1, 2, 4, 5, 7, 8\}$
  - $BN(B)(A_2) = \{1, 2, 4, 5, 7, 8\}$
- If $A_1 = \{1, 3, 5\}$ and $A_2 = \{2, 3, 4, 6\}$ then
  - $BN(B)(A_1) = \{1, 2, 4, 5, 7, 8\}$
  - $BN(B)(A_2) = \{1, 2, 4, 5, 7, 8\}$

Many-valued contexts

Indiscernibility

Given a complete many-valued context (Ob, At, Va, I), a nonempty subset of objects $A \subseteq Ob$ and a subset of attributes $B \subseteq At$, let

- $\alpha(B)(A) = \frac{Card(\underline{\underline{IND(B)}}(A))}{Card(\overline{\overline{IND(B)}}(A))}$

be the $IND(B)$-accuracy measure of A

Obviously

- $0 \leqslant \alpha(B)(A) \leqslant 1$

Moreover

- A is $IND(B)$-definable iff $\alpha(B)(A) = 1$
- A is $IND(B)$-rough iff $\alpha(B)(A) < 1$

Given a complete many-valued context (Ob, At, Va, I), a nonempty subset of objects $A \subseteq Ob$ and a subset of attributes $B \subseteq At$, let

- $\rho(B)(A) = \frac{Card(BN(B)(A))}{Card(\overline{IND(B)}(A))}$

be the $IND(B)$-roughness measure of A

Obviously

- $0 \leqslant \rho(B)(A) \leqslant 1$

Moreover

- A is $IND(B)$-definable iff $\rho(B)(A) = 0$
- A is $IND(B)$-rough iff $\rho(B)(A) > 0$

Many-valued contexts
Indiscernibility

Example 13:

- Suppose we are given a complete many-valued context (Ob, At, Va, I) where $Ob = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and let $B \subseteq At$ be a subset of attributes defining an equivalence relation $IND(B)$ with the following equivalence classes:
  - $\{1, 4, 8\}$
  - $\{2, 5, 7\}$
  - $\{3\}$
  - $\{6\}$
- If $A = \{1, 4, 5\}$ then
  - $\underline{IND(B)}(A) = \emptyset$
  - $BN(B)(A) = \{1, 2, 4, 5, 7, 8\}$
  - $\overline{IND(B)}(A) = \{1, 2, 4, 5, 7, 8\}$
  - $\alpha(B)(A) = \frac{0}{6} = 0.00$
  - $\rho(B)(A) = \frac{6}{6} = 1.00$

Many-valued contexts

Indiscernibility

Example 14:

- Suppose we are given a complete many-valued context (Ob, At, Va, I) where $Ob = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and let $B \subseteq At$ be a subset of attributes defining an equivalence relation $IND(B)$ with the following equivalence classes:
  - $\{1, 4, 8\}$
  - $\{2, 5, 7\}$
  - $\{3\}$
  - $\{6\}$
- If $A = \{3, 5\}$ then
  - $\underline{IND(B)}(A) = \{3\}$
  - $BN(B)(A) = \{2, 5, 7\}$
  - $\overline{IND(B)}(A) = \{2, 3, 5, 7\}$
  - $\alpha(B)(A) = \frac{1}{4} = 0.25$
  - $\rho(B)(A) = \frac{3}{4} = 0.75$

Example 15:

- Suppose we are given a complete many-valued context (Ob, At, Va, I) where $Ob = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and let $B \subseteq At$ be a subset of attributes defining an equivalence relation $IND(B)$ with the following equivalence classes:
  - $\{1, 4, 8\}$
  - $\{2, 5, 7\}$
  - $\{3\}$
  - $\{6\}$
- If $A = \{3, 6, 8\}$ then
  - $\underline{IND(B)}(A) = \{3, 6\}$
  - $BN(B)(A) = \{1, 4, 8\}$
  - $\overline{IND(B)}(A) = \{1, 3, 4, 6, 8\}$
  - $\alpha(B)(A) = \frac{2}{5} = 0.40$
  - $\rho(B)(A) = \frac{3}{5} = 0.60$

Given a complete many-valued context (Ob, At, Va, I), a nonempty family of nonempty subset of objects $F = \{A_1, \ldots, A_n\}$ and a subset of attributes $B \subseteq At$, let

- $\alpha(B)(F) = \frac{\Sigma_i Card(\underline{IND(B)}(A_i))}{\Sigma_i Card(\overline{IND(B)}(A_i))}$

be the $IND(B)$-accuracy of approximation of F and let

- $\gamma(B)(F) = \frac{\Sigma_i Card(\underline{IND(B)}(A_i))}{Card(Ob)}$

be the $IND(B)$-quality of approximation of F

Many-valued contexts

Indiscernibility

Given a complete many-valued context (Ob, At, Va, I), subsets of objects $A_1, A_2 \subseteq Ob$ and a subset of attributes $B \subseteq At$, let

- $A_1\ \underline{sim(B)}\ A_2$ iff $\underline{IND(B)}(A_1) = \underline{IND(B)}(A_2)$
- $A_1\ \overline{sim(B)}\ A_2$ iff $\overline{IND(B)}(A_1) = \overline{IND(B)}(A_2)$

Intuitive reading

- $A_1\ \underline{sim(B)}\ A_2$: "the positive examples of A_1 and A_2 are the same"
- $A_1\ \overline{sim(B)}\ A_2$: "the negative examples of A_1 and A_2 are the same"

Obviously $\underline{sim(B)}$ and $\overline{sim(B)}$ are equivalence relations

Many-valued contexts

Indiscernibility

Example 16:

- Suppose we are given a complete many-valued context (Ob, At, Va, I) where $Ob = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and let $B \subseteq At$ be a subset of attributes defining an equivalence relation $IND(B)$ with the following equivalence classes:
  - $\{1, 4, 5\}$
  - $\{2, 3\}$
  - $\{6\}$
  - $\{7, 8\}$
- If $A_1 = \{1, 2, 3\}$ and $A_2 = \{2, 3, 7\}$ then
  - $\underline{IND(B)}(A_1) = \{2, 3\}$
  - $\underline{IND(B)}(A_2) = \{2, 3\}$
- If $A_1 = \{1, 2, 7\}$ and $A_2 = \{2, 3, 4, 8\}$ then
  - $\overline{IND(B)}(A_1) = \{1, 2, 3, 4, 5, 7, 8\}$
  - $\overline{IND(B)}(A_2) = \{1, 2, 3, 4, 5, 7, 8\}$

Proposition 14:

1. $A_1 \ \underline{sim(B)}\ A_2$ iff $(A_1 \cap A_2)\ \underline{sim(B)}\ A_1$ and $(A_1 \cap A_2)\ \underline{sim(B)}\ A_2$
2. if $A_1\ \underline{sim(B)}\ A_1'$ and $A_2\ \underline{sim(B)}\ A_2'$ then $(A_1 \cap A_2)\ \underline{sim(B)}\ (A_1' \cap A_2')$
3. if $A_1\ \underline{sim(B)}\ A_2$ then $(A_1 \cap (Ob \setminus A_2))\ \underline{sim(B)}\ \emptyset$
4. if $A_1 \subseteq A_2$ then $A_2\ \underline{sim(B)}\ \emptyset$ implies $A_1\ \underline{sim(B)}\ \emptyset$
5. if $A_1 \subseteq A_2$ then $A_1\ \underline{sim(B)}\ Ob$ implies $A_2\ \underline{sim(B)}\ Ob$
6. if $A_1\ \underline{sim(B)}\ \emptyset$ or $A_2\ \underline{sim(B)}\ \emptyset$ then $(A_1 \cap A_2)\ \underline{sim(B)}\ \emptyset$

Many-valued contexts

Indiscernibility

Proposition 15:

1. $A_1 \ \overline{sim(B)} \ A_2$ iff $(A_1 \cup A_2) \ \overline{sim(B)} \ A_1$ and $(A_1 \cup A_2) \ \overline{sim(B)} \ A_2$
2. if $A_1 \ \overline{sim(B)} \ A_1'$ and $A_2 \ \overline{sim(B)} \ A_2'$ then $(A_1 \cup A_2) \ \overline{sim(B)} \ (A_1' \cup A_2')$
3. if $A_1 \ \overline{sim(B)} \ A_2$ then $(A_1 \cup (Ob \setminus A_2)) \ \overline{sim(B)} \ Ob$
4. if $A_1 \subseteq A_2$ then $A_2 \ \overline{sim(B)} \ \emptyset$ implies $A_1 \ \overline{sim(B)} \ \emptyset$
5. if $A_1 \subseteq A_2$ then $A_1 \ \overline{sim(B)} \ Ob$ implies $A_2 \ \overline{sim(B)} \ Ob$
6. if $A_1 \ \overline{sim(B)} \ Ob$ or $A_2 \ \overline{sim(B)} \ Ob$ then $(A_1 \cup A_2) \ \overline{sim(B)} \ Ob$

Proposition 16:

1. $\underline{IND(B)}(A)$ is the intersection of all subsets of objects $A' \subseteq Ob$ such that $A\ \underline{sim(B)}\ A'$
2. $\overline{IND(B)}(A)$ is the union of all subsets of objects $A' \subseteq Ob$ such that $A\ \overline{sim(B)}\ A'$

Many-valued contexts

Ternary contexts

Ternary context: structure of the form $\mathcal{S} = (Ob, At, Co, I)$ where

- *Ob* is a nonempty set of formal objects
- *At* is a nonempty set of formal attributes
- *Co* is a nonempty set of formal conditions
- *I* is a ternary relation between *Ob*, *At* and *Co*

Ternary contexts will usually be denoted

- $\mathcal{S} = (S_1, S_2, S_3, I)$

Many-valued contexts

Ternary contexts

A ternary context can be represented by a cross cube where

- 1-rows are headed by object names (X, Y, etc)
- 2-rows are headed by attribute names (x, y, etc)
- 3-rows are headed by condition names (α, β, etc)

A cross in 1-row X, 2-row x and 3-row α means that

- the object X has the attribute x under the condition α

Many-valued contexts

Ternary contexts

Given a ternary context $\mathcal{S} = (S_1, S_2, S_3, I)$, $i, j, k \in \{1, 2, 3\}$ pairwise distinct, a set $A_i \subseteq S_i$ of S_i-elements and a set $A_j \subseteq S_j$ of S_j-elements, we define

- $(A_i, A_j)^k = \{x_k \in S_k \colon I(x_i, x_j, x_k)$ for every $x_i \in A_i$ and for every $x_j \in A_j\}$

i.e. the set of S_k-elements common to the pairs (x_i, x_j) in $A_i \times A_j$

It is still a problem to generalize to ternary concepts the techniques in formal concept analysis that are presented in these slides

A ternary concept of the ternary context (S_1, S_2, S_3, I) is a triple (A_1, A_2, A_3) with

- $A_1 \subseteq S_1$
- $A_2 \subseteq S_2$
- $A_3 \subseteq S_3$
- $(A_1, A_2)^3 = A_3$
- $(A_1, A_3)^2 = A_2$
- $(A_2, A_3)^1 = A_1$

We call

- A_1 the extent of the concept (A_1, A_2, A_3)
- A_2 the intent of the concept (A_1, A_2, A_3)
- A_3 the mode of the concept (A_1, A_2, A_3)

Determination and representation

Determination and representation

A context for the planets

	small	*medium*	*large*	*near*	*far*	*yes*	*no*
Mercury	⊗			⊗			⊗
Venus	⊗			⊗			⊗
Earth	⊗			⊗		⊗	
Mars	⊗			⊗		⊗	
Jupiter			⊗		⊗	⊗	
Saturn			⊗		⊗	⊗	
Uranus		⊗			⊗	⊗	
Neptune		⊗			⊗	⊗	
Pluto	⊗				⊗	⊗	

Determination and representation

Contexts and concepts

Formal context: structure of the forme $\mathcal{K} = (Ob, At, I)$ where

- Ob is a nonempty set of formal objects
- At is a nonempty set of formal attributes
- I is a binary relation between Ob and At

Within the context of the planets

- $Ob = \{Mercury, Venus, \ldots\}$
- $At = \{small, medium, \ldots\}$
- $I = \{(Mercury, small), (Mercury, near), \ldots\}$

For a set $A \subseteq Ob$ of objects, we define

- $A' = \{x \in At\colon X \; I \; x \text{ for every } X \in A\}$

i.e. the set of attributes common to the objects in A

For a set $B \subseteq At$ of attributes, we define

- $B' = \{X \in Ob\colon X \; I \; x \text{ for every } x \in B\}$

i.e. the set of objects which have all attributes in B

Within the context of the planets

- $\{Earth, Mars\}' = \{small, near, yes\}$
- $\{small, near\}' = \{Mercury, Venus, Earth, Mars\}$

A formal concept of the context (Ob, At, I) is a pair (A, B) with

- $A \subseteq Ob$
- $B \subseteq At$
- $A' = B$
- $B' = A$

Within the context of the planets

- $(\{Earth, Mars\}, \{small, near, yes\})$
- $(\{Mercury, Venus, Earth, Mars\}, \{small, near\})$

Determination and representation

The ordering of concepts

If (A_1, B_1) and (A_2, B_2) are concepts of a context then

- $A_1 \subseteq A_2$ iff $B_2 \subseteq B_1$

If $A_1 \subseteq A_2$ and $B_2 \subseteq B_1$ then we say that

- (A_1, B_1) is a subconcept of (A_2, B_2)
- (A_2, B_2) is a superconcept of (A_1, B_1)

and we write

- $(A_1, B_1) \leqslant (A_2, B_2)$

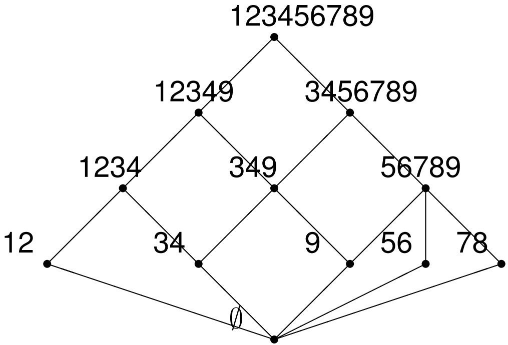
The set of all concepts of (Ob, At, I) ordered in this way

- is denoted by $\underline{\mathcal{B}}(Ob, At, I)$
- is called the concept lattice of the context (Ob, At, I)

Determination and representation

The ordering of concepts

Within the context of the planets (*Mercury* $= 1$, *Venus* $= 2$, *Earth* $= 3$, *Mars* $= 4$, *Jupiter* $= 5$, *Saturn* $= 6$, *Uranus* $= 7$, *Neptune* $= 8$ et *Pluto* $= 9$)



Determination and representation

The ordering of concepts

Theorem 3: The concept lattice $\underline{\mathcal{B}}(Ob, At, I)$ is a complete lattice in which infimum and supremum are given by

- $\bigwedge_{t \in T}(A_t, B_t) = (\bigcap_{t \in T} A_t, (\bigcup_{t \in T} B_t)'')$
- $\bigvee_{t \in T}(A_t, B_t) = ((\bigcup_{t \in T} A_t)'', \bigcap_{t \in T} B_t)$

A simple-minded and extremely inefficient way of determining all the concepts of a context $\mathcal{K} = (Ob, At, I)$

1. choose a set A of objects
2. compute the set A' of attributes common to the objects in A
3. compute the set A'' of objects which have all attributes in A'

Then the pair (A'', A') is a concept

Determination and representation

The determination problem

The concept $(\{Earth, Mars\}, \{small, near, yes\})$

	small	*medium*	*large*	*near*	*far*	*yes*	*no*
Mercury	$\otimes$			$\otimes$			$\otimes$
Venus	$\otimes$			$\otimes$			$\otimes$
Earth	$\otimes$			$\otimes$		$\otimes$	
Mars	$\otimes$			$\otimes$		$\otimes$	
Jupiter			$\otimes$		$\otimes$	$\otimes$	
Saturn			$\otimes$		$\otimes$	$\otimes$	
Uranus		$\otimes$			$\otimes$	$\otimes$	
Neptune		$\otimes$			$\otimes$	$\otimes$	
Pluto	$\otimes$				$\otimes$	$\otimes$	

The concept $(\{Earth, Mars\}, \{small, near, yes\})$

	small	*medium*	*large*	*near*	*far*	*yes*	*no*
Mercury	⊗			⊗			⊗
Venus	⊗			⊗			⊗
Earth	⊗			⊗		⊗	
Mars	⊗			⊗		⊗	
Jupiter			⊗		⊗	⊗	
Saturn			⊗		⊗	⊗	
Uranus		⊗			⊗	⊗	
Neptune		⊗			⊗	⊗	
Pluto	⊗				⊗	⊗	

Determination and representation

The determination problem

The concept $(\{Earth, Mars\}, \{small, near, yes\})$

	small	*medium*	*large*	*near*	*far*	*yes*	*no*
Mercury	$\otimes$			$\otimes$			$\otimes$
Venus	$\otimes$			$\otimes$			$\otimes$
Earth	$\otimes$			$\otimes$		$\otimes$	
Mars	$\otimes$			$\otimes$		$\otimes$	
Jupiter			$\otimes$		$\otimes$	$\otimes$	
Saturn			$\otimes$		$\otimes$	$\otimes$	
Uranus		$\otimes$			$\otimes$	$\otimes$	
Neptune		$\otimes$			$\otimes$	$\otimes$	
Pluto	$\otimes$				$\otimes$	$\otimes$	

The concept $(\{Mercury, Venus, Earth, Mars\}, \{small, near\})$

	small	*medium*	*large*	*near*	*far*	*yes*	*no*
Mercury	⊗			⊗			⊗
Venus	⊗			⊗			⊗
Earth	⊗			⊗		⊗	
Mars	⊗			⊗		⊗	
Jupiter			⊗		⊗	⊗	
Saturn			⊗		⊗	⊗	
Uranus		⊗			⊗	⊗	
Neptune		⊗			⊗	⊗	
Pluto	⊗				⊗	⊗	

The concept $(\{Mercury, Venus, Earth, Mars\}, \{small, near\})$

	small	*medium*	*large*	*near*	*far*	*yes*	*no*
Mercury	⊗			⊗			⊗
Venus	⊗			⊗			⊗
Earth	⊗			⊗		⊗	
Mars	⊗			⊗		⊗	
Jupiter			⊗		⊗	⊗	
Saturn			⊗		⊗	⊗	
Uranus		⊗			⊗	⊗	
Neptune		⊗			⊗	⊗	
Pluto	⊗				⊗	⊗	

The concept ({*Mercury*, *Venus*, *Earth*, *Mars*}, {*small*, *near*})

	small	*medium*	*large*	*near*	*far*	*yes*	*no*
Mercury	⊗			⊗			⊗
Venus	⊗			⊗			⊗
Earth	⊗			⊗		⊗	
Mars	⊗			⊗		⊗	
Jupiter			⊗		⊗	⊗	
Saturn			⊗		⊗	⊗	
Uranus		⊗			⊗	⊗	
Neptune		⊗			⊗	⊗	
Pluto	⊗				⊗	⊗	

Determination and representation

An algorithm for finding all concepts of a given context

A simple-minded and extremely inefficient way of determining all the concepts of a context $\mathcal{K} = (Ob, At, I)$

1. choose a set B of attributes
2. compute the set B' of objects which have all attributes in B
3. compute the set B'' of attributes common to the objects in B'

Then the pair (B', B'') is a concept

Remark that for all $A \subseteq Ob$ and for all $B \subseteq At$

- $A' = \bigcap_{X \in A} X'$ and $B' = \bigcap_{x \in B} x'$

In particular, if (A, B) is a concept then

- $A = \bigcap_{x \in B} x'$ and $B = \bigcap_{X \in A} X'$

Determination and representation

An algorithm for finding all concepts of a given context

Let $\mathcal{K} = (Ob, At, I)$ be a given context

Determination and representation

An algorithm for finding all concepts of a given context

Example 32:

	a	b	c	d	e	f	g
S	⊗	⊗		⊗		⊗	
T	⊗	⊗		⊗	⊗		
U	⊗	⊗		⊗	⊗	⊗	⊗
V	⊗		⊗		⊗	⊗	
W		⊗		⊗			
X	⊗					⊗	

Let $\mathcal{K} = (Ob, At, I)$ be a given context

1. draw up a table with two columns headed **Attributes** (**A**) and **Extents** (**E**), leave the first cell of the **A** column empty and write *Ob* in the first cell of the **E** column

Example 32:

	a	*b*	*c*	*d*	*e*	*f*	*g*
S	⊗	⊗		⊗		⊗	
T	⊗	⊗		⊗	⊗		
U	⊗	⊗		⊗	⊗	⊗	⊗
V	⊗		⊗		⊗	⊗	
W		⊗		⊗			
X	⊗					⊗	

A	**E**
	Ob

Determination and representation

An algorithm for finding all concepts of a given context

Let $\mathcal{K} = (Ob, At, I)$ be a given context

1. draw up a table with two columns headed **Attributes** (**A**) and **Extents** (**E**), leave the first cell of the **A** column empty and write *Ob* in the first cell of the **E** column
2. find a maximal attribute extent, say x'

Example 32:

	a	*b*	*c*	*d*	*e*	*f*	*g*
S	⊗	⊗		⊗		⊗	
T	⊗	⊗		⊗	⊗		
U	⊗	⊗		⊗	⊗	⊗	⊗
V	⊗		⊗		⊗	⊗	
W		⊗		⊗			
X	⊗					⊗	

A	**E**
	Ob

Determination and representation

An algorithm for finding all concepts of a given context

Let $\mathcal{K} = (Ob, At, I)$ be a given context

1. draw up a table with two columns headed **Attributes** (**A**) and **Extents** (**E**), leave the first cell of the **A** column empty and write *Ob* in the first cell of the **E** column
2. find a maximal attribute extent, say x'
   - 2.1 if the set x' is not already in the **E** column, add the row $[x, x']$ to the table, intersect the set x' with all previous extents in **E** , add these intersections to the **E** column unless they are already in the list
   - 2.2 if the set x' is already in the **E** column, add the label x to the attribute cell of the rwo where x' previously occured

Example 32:

	a	*b*	*c*	*d*	*e*	*f*	*g*
S	⊗	⊗		⊗		⊗	
T	⊗	⊗		⊗	⊗		
U	⊗	⊗		⊗	⊗	⊗	⊗
V	⊗		⊗		⊗	⊗	
W		⊗		⊗			
X	⊗					⊗	

A	**E**
	Ob
a	*STUVX*

Determination and representation

An algorithm for finding all concepts of a given context

Let $\mathcal{K} = (Ob, At, I)$ be a given context

1. draw up a table with two columns headed **Attributes** (**A**) and **Extents** (**E**), leave the first cell of the **A** column empty and write *Ob* in the first cell of the **E** column
2. find a maximal attribute extent, say x'
   - 2.1 if the set x' is not already in the **E** column, add the row $[x, x']$ to the table, intersect the set x' with all previous extents in **E** , add these intersections to the **E** column unless they are already in the list
   - 2.2 if the set x' is already in the **E** column, add the label x to the attribute cell of the rwo where x' previously occured
3. delete the column below x from the context
4. if the last column has been deleted, stop, otherwise return to 2

Determination and representation

An algorithm for finding all concepts of a given context

Example 32:

	a	*b*	*c*	*d*	*e*	*f*	*g*
S	⊗	⊗		⊗		⊗	
T	⊗	⊗		⊗	⊗		
U	⊗	⊗		⊗	⊗	⊗	⊗
V	⊗		⊗		⊗	⊗	
W		⊗		⊗			
X	⊗					⊗	

A	**E**
	Ob
a	*STUVX*
b	*STUW*

Determination and representation

An algorithm for finding all concepts of a given context

Example 32:

	a	*b*	*c*	*d*	*e*	*f*	*g*
S	⊗	⊗		⊗		⊗	
T	⊗	⊗		⊗	⊗		
U	⊗	⊗		⊗	⊗	⊗	⊗
V	⊗		⊗		⊗	⊗	
W		⊗		⊗			
X	⊗					⊗	

A	**E**
	Ob
a	*STUVX*
b	*STUW*
	STU

Determination and representation

An algorithm for finding all concepts of a given context

Example 32:

	a	*b*	*c*	*d*	*e*	*f*	*g*
S	⊗	⊗		⊗		⊗	
T	⊗	⊗		⊗	⊗		
U	⊗	⊗		⊗	⊗	⊗	⊗
V	⊗		⊗		⊗	⊗	
W		⊗		⊗			
X	⊗					⊗	

A	**E**
	Ob
a	*STUVX*
bd	*STUW*
	STU

Determination and representation

An algorithm for finding all concepts of a given context

Example 32:

	a	*b*	*c*	*d*	*e*	*f*	*g*
S	⊗	⊗		⊗		⊗	
T	⊗	⊗		⊗	⊗		
U	⊗	⊗		⊗	⊗	⊗	⊗
V	⊗		⊗		⊗	⊗	
W		⊗		⊗			
X	⊗					⊗	

A	**E**
	Ob
a	*STUVX*
bd	*STUW* *STU*
f	*SUVX*

Determination and representation

An algorithm for finding all concepts of a given context

Example 32:

	a	*b*	*c*	*d*	*e*	*f*	*g*
S	$\otimes$	$\otimes$		$\otimes$		$\otimes$	
T	$\otimes$	$\otimes$		$\otimes$	$\otimes$		
U	$\otimes$	$\otimes$		$\otimes$	$\otimes$	$\otimes$	$\otimes$
V	$\otimes$		$\otimes$		$\otimes$	$\otimes$	
W		$\otimes$		$\otimes$			
X	$\otimes$					$\otimes$	

A	**E**
	Ob
a	*STUVX*
bd	*STUW*
	STU
f	*SUVX*
	SU

Determination and representation

An algorithm for finding all concepts of a given context

Example 32:

	a	*b*	*c*	*d*	*e*	*f*	*g*
S	⊗	⊗		⊗		⊗	
T	⊗	⊗		⊗	⊗		
U	⊗	⊗		⊗	⊗	⊗	⊗
V	⊗		⊗		⊗	⊗	
W		⊗		⊗			
X	⊗					⊗	

A	**E**
	Ob
a	*STUVX*
bd	*STUW* *STU*
f	*SUVX* *SU*
e	*TUV*

Determination and representation

An algorithm for finding all concepts of a given context

Example 32:

	a	b	c	d	e	f	g
S	⊗	⊗		⊗		⊗	
T	⊗	⊗		⊗	⊗		
U	⊗	⊗		⊗	⊗	⊗	⊗
V	⊗		⊗		⊗	⊗	
W		⊗		⊗			
X	⊗					⊗	

A	**E**
	Ob
a	*STUVX*
bd	*STUW*
	STU
f	*SUVX*
	SU
e	*TUV*
	TU
	UV
	U

Determination and representation

An algorithm for finding all concepts of a given context

Example 32:

	a	*b*	*c*	*d*	*e*	*f*	*g*
S	⊗	⊗		⊗		⊗	
T	⊗	⊗		⊗	⊗		
U	⊗	⊗		⊗	⊗	⊗	⊗
V	⊗		⊗		⊗	⊗	
W		⊗		⊗			
X	⊗					⊗	

A	**E**
	Ob
a	*STUVX*
bd	*STUW*
	STU
f	*SUVX*
	SU
e	*TUV*
	TU
	UV
	U
c	*V*

Determination and representation

An algorithm for finding all concepts of a given context

Example 32:

	a	*b*	*c*	*d*	*e*	*f*	*g*
S	$\otimes$	$\otimes$		$\otimes$		$\otimes$	
T	$\otimes$	$\otimes$		$\otimes$	$\otimes$		
U	$\otimes$	$\otimes$		$\otimes$	$\otimes$	$\otimes$	$\otimes$
V	$\otimes$		$\otimes$		$\otimes$	$\otimes$	
W		$\otimes$		$\otimes$			
X	$\otimes$					$\otimes$	

A	**E**
	Ob
a	*STUVX*
bd	*STUW*
	STU
f	*SUVX*
	SU
e	*TUV*
	TU
	UV
	U
c	*V*
	$\emptyset$

Determination and representation

An algorithm for finding all concepts of a given context

Example 32:

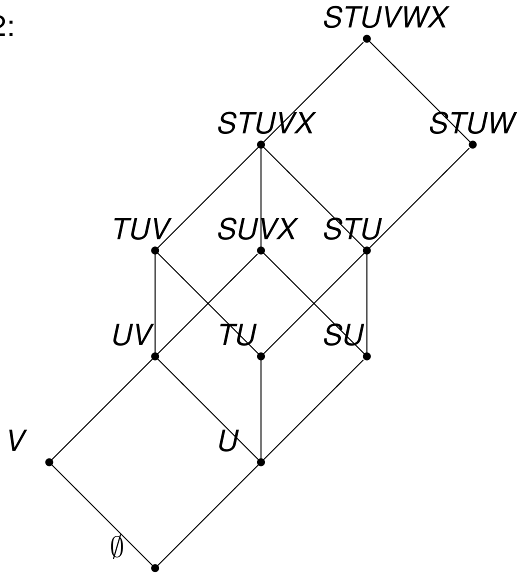
	a	b	c	d	e	f	g
S	⊗	⊗		⊗		⊗	
T	⊗	⊗		⊗	⊗		
U	⊗	⊗		⊗	⊗	⊗	⊗
V	⊗		⊗		⊗	⊗	
W		⊗		⊗			
X	⊗					⊗	

A	**E**
	Ob
a	*STUVX*
bd	*STUW*
	STU
f	*SUVX*
	SU
eg	*TUV*
	TU
	UV
	U
c	*V*
	$\emptyset$

Determination and representation

An algorithm for finding all concepts of a given context

Example 32:



It is still possible to effect improvements in finding all concepts of a given context

- Choi, V.: *Faster algorithms for constructing a concept (Galois) lattice.* In Butenko, S., Chaovalitwongse, W., Pardalos, P. (Editors): *Clustering Challenges in Biological Networks.* World Scientific (2009) 169–185.
- Kuznetsov, S., Obiedkov, S.: *Comparing performance of algorithms for generating concept lattices.* Journal of Experimental & Theoretical Artificial Intelligence **14** (2002) 189–216.

It is still possible to effect improvements in finding all concepts of a given context

- Van der Merwe, D., Obiedkov, S., Kourie, D.: *AddIntent: a new incremental algorithm for constructing concept lattices.* In Eklund, P. (Editor): *ICFCA 2004.* Springer-Verlag (2004) 372–385.
- Valtchev, P., Missaoui, R.: *Building concept (Galois) lattices from parts: generalizing the incremental methods.* In Delugach, H., Stumme, G. (Editors): *ICCS 2001.* Springer-Verlag (2001) 290–303.

Determination and representation

Drawing the concept lattice of a given context

Given a formal context $\mathbb{K} = (Ob, At, I)$, the problem is to arrange the nodes and lines of the diagram of its concept lattice in order to achieve

- the best visual quality
- the best visual readability

Do it fast and automatically

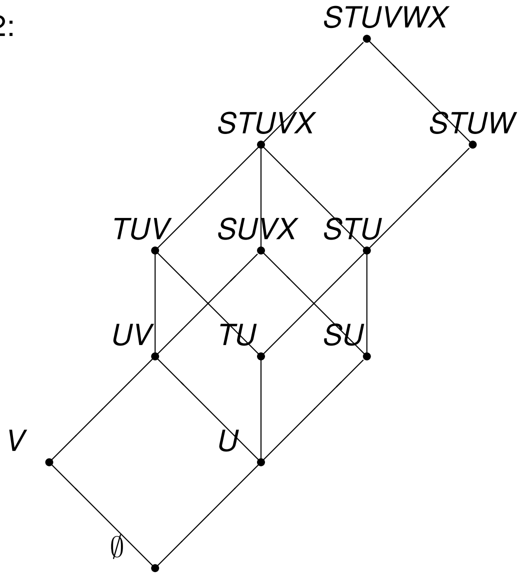
Example 32:

	a	b	c	d	e	f	g
S	$\otimes$	$\otimes$		$\otimes$		$\otimes$	
T	$\otimes$	$\otimes$		$\otimes$	$\otimes$		
U	$\otimes$	$\otimes$		$\otimes$	$\otimes$	$\otimes$	$\otimes$
V	$\otimes$		$\otimes$		$\otimes$	$\otimes$	
W		$\otimes$		$\otimes$			
X	$\otimes$					$\otimes$	

Determination and representation

Drawing the concept lattice of a given context

Example 32:



Determination and representation

Drawing the concept lattice of a given context

There are several subjective human æsthetics criteria

- minimizing line crossings (planarity)
- maximizing angle between incident lines
- maximizing symmetries
- maximizing compactness

These criteria are often contradictory and lead to computationaly difficult (NP-complete) problems

How large lattices one can draw by a computer?

- Up to about a hundred of nodes

Determination and representation

A force directed approach for drawing the concept lattice of a given context

Let $\mathcal{K} = (Ob, At, I)$ be a given context

1. within a 3-dimensional space, organize nodes of the concept lattice in layers based on their distance from the top node $(\emptyset', \emptyset'')$
2. for each layer, randomly arrange its nodes as the vertices of a regular polygon which has a circumscribed circle of radius 1
3. between each pair of nodes occurring in two successive layers, calculate imaginary repulsive and attractive forces depending on how much this pair of nodes overlap
4. inside each layer, modify the positions of its nodes according to the forces calculated in step 3
5. if the resulting diagram is not "good enough" then go to step 3

Determination and representation

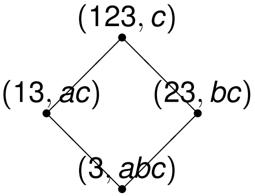
A vectorial approach for drawing the concept lattice of a given context

Let $\mathcal{K} = (Ob, At, I)$ be a given context

1. choose a point $pos_0 \in \mathbb{R} \times \mathbb{R}$
2. associate to each object $X \in Ob$ a vector $\vec{vec}(X) \in \mathbb{R} \times \mathbb{R}^{+\star}$
3. for each extent A of a $\mathcal{K}$-concept, compute $pos_0 + \Sigma\{\vec{vec}(X) : X \in A\}$

Example 33:

	a	b	c
1	$\otimes$		$\otimes$
2		$\otimes$	$\otimes$
3	$\otimes$	$\otimes$	$\otimes$



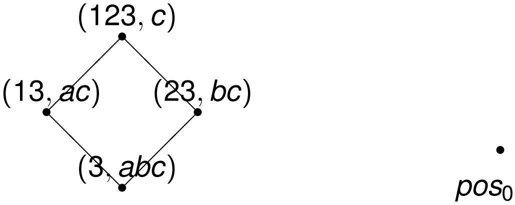
Determination and representation

A vectorial approach for drawing the concept lattice of a given context

Example 33:

- choose a point $pos_0 \in \mathbb{R} \times \mathbb{R}$

	a	b	c
1	$\otimes$		$\otimes$
2		$\otimes$	$\otimes$
3	$\otimes$	$\otimes$	$\otimes$

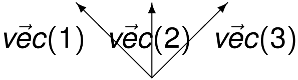


Determination and representation

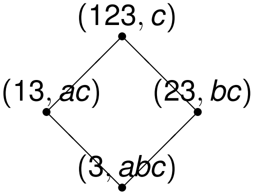
A vectorial approach for drawing the concept lattice of a given context

Example 33:

- associate to each object $X \in Ob$ a vector $\vec{vec}(X) \in \mathbb{R} \times \mathbb{R}^{+\star}$



	a	b	c
1	$\otimes$		$\otimes$
2		$\otimes$	$\otimes$
3	$\otimes$	$\otimes$	$\otimes$



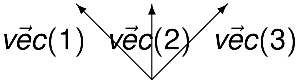
Determination and representation

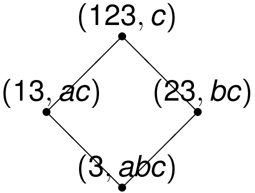
A vectorial approach for drawing the concept lattice of a given context

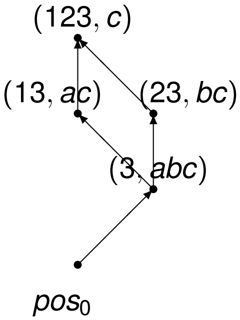
Example 33:

- for each extent A of a $\mathcal{K}$-concept, compute $pos_0 + \Sigma\{\vec{vec}(X) : X \in A\}$

	a	b	c
1	$\otimes$		$\otimes$
2		$\otimes$	$\otimes$
3	$\otimes$	$\otimes$	$\otimes$







Determination and representation

A dichotomic approach for drawing the concept lattice of a given context

Let $\mathcal{K} = (Ob, At, I)$ be a given context

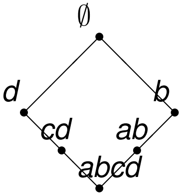
1. choose $At_1 \subseteq At$ and $At_2 \subseteq At$ such that $At_1 \cup At_2 = At$
2. draw the concept lattices of the contexts
   $\mathcal{K}_1 = (Ob, At_1, I \cap (Ob \times At_1))$ and
   $\mathcal{K}_2 = (Ob, At_2, I \cap (Ob \times At_2))$
3. draw the product of these lattices
4. for each $\mathcal{K}$-intent B, compute the corresponding element $(B \cap At_1, B \cap At_2)$ in the product

Determination and representation

A dichotomic approach for drawing the concept lattice of a given context

Example 34:

	a	*b*	*c*	*d*
1		⊗		
2	⊗	⊗		
3				⊗
4			⊗	⊗



Determination and representation

A dichotomic approach for drawing the concept lattice of a given context

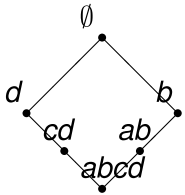
Example 34:

- choose $At_1 \subseteq At$ and $At_2 \subseteq At$ such that $At_1 \cup At_2 = At$

	a	b	c	d
1		⊗		
2	⊗	⊗		
3				⊗
4			⊗	⊗

	a	b
1		⊗
2	⊗	⊗
3		
4		

	c	d
1		
2		
3		⊗
4	⊗	⊗



Determination and representation

A dichotomic approach for drawing the concept lattice of a given context

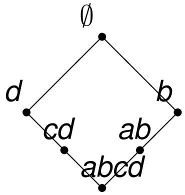
Example 34:

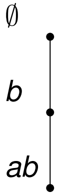
- draw the concept lattices of the contexts
  $\mathcal{K}_1 = (Ob, At_1, I \cap (Ob \times At_1))$ and
  $\mathcal{K}_2 = (Ob, At_2, I \cap (Ob \times At_2))$

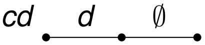
	a	b	c	d
1		⊗		
2	⊗	⊗		
3				⊗
4			⊗	⊗

	a	b
1		⊗
2	⊗	⊗
3		
4		

	c	d
1		
2		
3		⊗
4	⊗	⊗







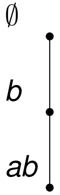
Determination and representation

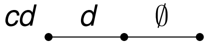
A dichotomic approach for drawing the concept lattice of a given context

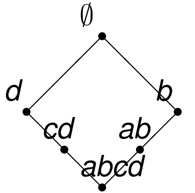
Example 34:

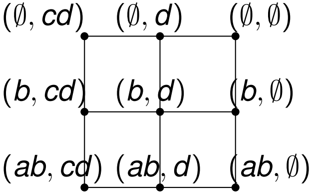
- draw the product of these lattices

	a	b	c	d
1		$\otimes$		
2	$\otimes$	$\otimes$		
3				$\otimes$
4			$\otimes$	$\otimes$









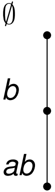
Determination and representation

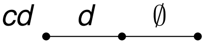
A dichotomic approach for drawing the concept lattice of a given context

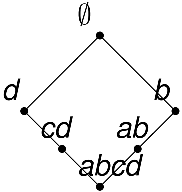
Example 34:

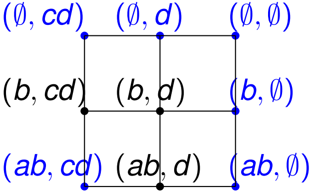
- for each $\mathcal{K}$-intent B, compute the corresponding element $(B \cap At_1, B \cap At_2)$ in the product

	a	b	c	d
1		$\otimes$		
2	$\otimes$	$\otimes$		
3				$\otimes$
4			$\otimes$	$\otimes$









It is still possible to effect improvements in drawing the concept lattice of a given context

- Freese, R.: *Automated lattice drawing.* In Eklund, P. (Editor): *ICFCA 2004.* Springer-Verlag (2004) 112–127.
- Tilley, T.: *Tool support for FCA.* In Eklund, P. (Editor): *ICFCA 2004.* Springer-Verlag (2004) 104–111.

Determination and representation

Implications between attributes

It is

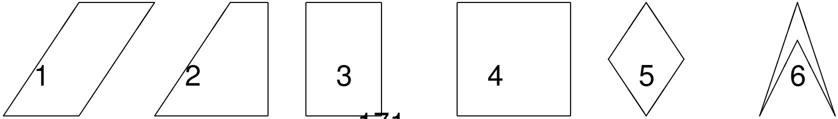
- often necessary to classify a large number of objects with respect to a relatively small number of attributes
- frequently useless or impracticable to write down the whole context

In such cases

- the concept lattice can be inferred from the implication between the attributes
- the concept lattice can be inferred from statements of the kind “every object with the attributes $x_1, y_1, \ldots$ also has the attributes $x_2, y_2, \ldots$”

Example 35:

	concave	*square*	*rectangle*	*equilateral*	*parallelogram*
1					×
2					
3			×		×
4		×	×	×	×
5				×	×
6	×				



Determination and representation

Implications between attributes

Implication between attributes in a given context $\mathcal{K} = (Ob, At, I)$

- implication $B_1 \longrightarrow B_2$ where B_1 and B_2 are sets of $\mathcal{K}$-attributes

Let B be a set of $\mathcal{K}$-attributes, $B_1 \longrightarrow B_2$ a $\mathcal{K}$-implication and $\mathcal{L}$ a set of $\mathcal{K}$-implications

- B respects $B_1 \longrightarrow B_2$ iff $B_1 \not\subseteq B$ or $B_2 \subseteq B$
- B respects $\mathcal{L}$ iff B respects every $\mathcal{K}$-implication $B_1 \longrightarrow B_2 \in \mathcal{L}$

Example 36:

	concave	*square*	*rectangle*	*equilateral*	*parallelogram*
1					×
2					
3			×		×
4		×	×	×	×
5				×	×
6	×				

$\{concave, parallelogram\} \longrightarrow \{square, rectangle, equilateral\}$
$\{square\} \longrightarrow \{rectangle, equilateral, parallelogram\}$
$\{rectangle\} \longrightarrow \{parallelogram\}$
$\{rectangle, equilateral, parallelogram\} \longrightarrow \{square\}$
$\{equilateral\} \longrightarrow \{parallelogram\}$

Determination and representation
Implications between attributes

Implication between attributes in a given context $\mathcal{K} = (Ob, At, I)$

- implication $B_1 \longrightarrow B_2$ where B_1 and B_2 are sets of $\mathcal{K}$-attributes

Let $B_1 \longrightarrow B_2$ a $\mathcal{K}$-implication and $\mathcal{L}$ a set of $\mathcal{K}$-implications

- $\mathcal{K}$ respects $B_1 \longrightarrow B_2$ iff B respects $B_1 \longrightarrow B_2$ for each $\mathcal{K}$-concept (A, B)
- $\mathcal{K}$ respects $\mathcal{L}$ iff $\mathcal{K}$ respect every $\mathcal{K}$-implication $B_1 \longrightarrow B_2 \in \mathcal{L}$

Implicational theory of $\mathcal{K}$

- Set $Imp(\mathcal{K})$ of all $\mathcal{K}$-implications that $\mathcal{K}$ respects

Determination and representation

Implications between attributes

Suppose that

- $\mathcal{K} = (Ob, At, I)$ is a context
- $B_1 \longrightarrow B_2$ is a $\mathcal{K}$-implication

Then the following conditions are equivalent

- $\mathcal{K}$ respects $B_1 \longrightarrow B_2$
- $B_1' \subseteq B_2'$
- $B_1'' \supseteq B_2$

Implicational closure of a set $\mathcal{L}$ of $\mathcal{K}$-implications : mapping $Cl_{\mathcal{L}}(\cdot) : 2^{At} \longrightarrow 2^{At}$ such that for all $B \subseteq At$, $Cl_{\mathcal{L}}(B)$ is the smallest set of $\mathcal{K}$-attributes containing B and respecting $\mathcal{L}$

Example 37:

	concave	*square*	*rectangle*	*equilateral*	*parallelogram*
1					×
2					
3			×		×
4		×	×	×	×
5				×	×
6	×				

If $\mathcal{L}$ contains the implications $\{rectangle\} \longrightarrow \{parallelogram\}$ and $\{rectangle, equilateral, parallelogram\} \longrightarrow \{square\}$ then

- $Cl_{\mathcal{L}}(\{rectangle, equilateral\}) =$ $\{square, rectangle, equilateral, parallelogram\}$

Let $B_1 \longrightarrow B_2$ be a $\mathcal{K}$-implication and $\mathcal{L}$ be a set of $\mathcal{K}$-implications

- $B_1 \longrightarrow B_2$ is a consequence of $\mathcal{L}$ iff $Cl_{\mathcal{L}}(B_1) \supseteq B_2$

Let $\mathcal{L}$ and $\mathcal{M}$ be sets of $\mathcal{K}$-implications

- $\mathcal{L}$ is sound for $\mathcal{M}$ iff every implication that follows from $\mathcal{L}$ is in $\mathcal{M}$
- $\mathcal{L}$ is complete for $\mathcal{M}$ iff every implication in $\mathcal{M}$ follows from $\mathcal{L}$

Let $\mathcal{L}$ be a set of $\mathcal{K}$-implications

- $\mathcal{L}$ is a base for $\mathcal{K}$ iff $\mathcal{L}$ is sound and complete for the set of all $\mathcal{K}$-implications that $\mathcal{K}$ respects
- $\mathcal{L}$ is a Duquenne-Guigues base for $\mathcal{K}$ iff $\mathcal{L}$ is a base for $\mathcal{K}$ that is of minimum cardinality

Example 38:

	concave	*square*	*rectangle*	*equilateral*	*parallelogram*
1					×
2					
3			×		×
4		×	×	×	×
5				×	×
6	×				

$\{concave, parallelogram\} \longrightarrow \{square, rectangle, equilateral\}$
$\{square\} \longrightarrow \{rectangle, equilateral, parallelogram\}$
$\{rectangle\} \longrightarrow \{parallelogram\}$
$\{rectangle, equilateral, parallelogram\} \longrightarrow \{square\}$
$\{equilateral\} \longrightarrow \{parallelogram\}$

Determination and representation

Implications between attributes

Suppose that

- $\mathcal{K} = (Ob, At, I)$ is a context
- $B \subseteq At$

Then B is a good attribute subset of $\mathcal{K}$ iff

- $B'' \supsetneq B$
- for all $C \subsetneq B$, if $C'' \supsetneq C$ then $B \supsetneq C''$

Example 39:

	concave	*square*	*rectangle*	*equilateral*	*parallelogram*
1					×
2					
3			×		×
4		×	×	×	×
5				×	×
6	×				

$\{concave, parallelogram\}$
$\{square\}$
$\{rectangle\}$
$\{rectangle, equilateral, parallelogram\}$
$\{equilateral\}$

Suppose that

- $\mathcal{K} = (Ob, At, I)$ is a context

Then

- $\{B \longrightarrow B'' : B \subseteq At$ is a good attribute subset of $\mathcal{K}\}$ is a Duquenne-Guigues base for $\mathcal{K}$

Example 40: Within the context of the quadrilaterals

- $\{concave, parallelogram\} \longrightarrow \{square, rectangle, equilateral\}$
- $\{square\} \longrightarrow \{rectangle, equilateral, parallelogram\}$
- $\{rectangle\} \longrightarrow \{parallelogram\}$
- $\{rectangle, equilateral, parallelogram\} \longrightarrow \{square\}$
- $\{equilateral\} \longrightarrow \{parallelogram\}$

We consider the following problem

- Deciding whether a set of attributes is a good attribute subset of a context

  Input: A context $\mathcal{K} = (Ob, At, I)$ and a set of attributes $B \subseteq At$

  Output: Decide whether B is a good attribute subset of $\mathcal{K}$

Suppose that

- $\mathcal{K} = (Ob, At, I)$ is a context
- $B \subseteq At$ is a set of $\mathcal{K}$-attributes

We shall say that

- B is closed iff $B'' = B$
- B is quasi-closed iff for all sets $C \subsetneq B$ of $\mathcal{K}$-attributes, $C'' \subseteq B$ or $C'' = B''$
- B is pseudo-closed iff B is not closed, B is quasi-closed and for all quasi-closed sets $C \subsetneq B$ of $\mathcal{K}$-attributes, $C'' \subsetneq B$

Note that

- if B is closed then B is quasi-closed

Determination and representation

Implications between attributes

Proposition 23: If $\mathcal{K} = (Ob, At, I)$ is a context and $B \subseteq At$ is a set of $\mathcal{K}$-attributes then

1. B is quasi-closed iff $B \cap C$ is closed for every closed set C with $B \not\subseteq C$
2. B is quasi-closed iff $B \cap X'$ is closed or $B \cap X' = B$ for any object $X \in Ob$
3. B is pseudo-closed iff B is a good attribute subset of $\mathcal{K}$

Proposition 24: If $\mathcal{K} = (Ob, At, I)$ is a context and $B_1, B_2 \subseteq At$ are sets of $\mathcal{K}$-attributes then

- if B_1, B_2 are quasi-closed then $B_1 \cap B_2$ is quasi-closed

Proposition 25: Testing whether $B \subseteq At$ is quasi-closed in the context $\mathcal{K} = (Ob, At, I)$ may be performed in $O(Card(Ob) \times Card(At))$ time

Proposition 26: The following problem is in *coNP*:

Input A context $\mathcal{K} = (Ob, At, I)$ and a set of attributes $B \subseteq At$

Output Decide whether B is a good attribute subset of $\mathcal{K}$

Determination and representation

Implications between attributes

A hypergraph $\mathcal{H} = (V, E)$ is a pair consisting of

- a finite nonempty set V (vertices)
- a set E of subsets of V (edges)

A hypergraph $\mathcal{H} = (V, E)$ is called simple iff

- none of $\mathcal{H}$'s edges contains another edge of $\mathcal{H}$

A hypergraph $\mathcal{H} = (V, E)$ is called saturated iff

- every subset of V is contained in at least one edge of $\mathcal{H}$ or it contains at least one edge of $\mathcal{H}$

Proposition 27: The following problem is *coNP*-complete:

Input A hypergraph $\mathcal{H} = (V, E)$

Output Decide whether $\mathcal{H}$ is saturated

Proposition 28: The following problem is in *coNP*:

Input A simple hypergraph $\mathcal{H} = (V, E)$

Output Decide whether $\mathcal{H}$ is saturated

A set of vertices $W \subseteq V$ is called a transversal of a hypergraph $\mathcal{H} = (V, E)$ iff

- W intersects every edge of $\mathcal{H}$

A set of vertices $W \subseteq V$ is called a minimal transversal of a hypergraph $\mathcal{H} = (V, E)$ iff

- W is a transversal of $\mathcal{H}$
- no proper subset of W is a transversal of $\mathcal{H}$

The set of all minimal transversals of a hypergraph $\mathcal{H} = (V, E)$ constitutes another hypergraph on V called the transversal of $\mathcal{H}$

Determination and representation

Implications between attributes

Proposition 28: The following problem is in *coNP*:

Input A simple hypergraph $\mathcal{H} = (V, E)$

Output Decide whether $\mathcal{H}$ is saturated

Proposition 29: The following problem is under polynomial transformations computationally equivalent to the problem considered in Proposition 28:

Input Two hypergraphs $\mathcal{G} = (V, E_G)$ and $\mathcal{H} = (V, E_H)$

Output Decide whether $\mathcal{G}$ is the transversal of $\mathcal{H}$

Proposition 30: The following problem is under polynomial transformations at least as hard as the problems considered in Proposition 28 and Proposition 29:

Input A context $\mathcal{K} = (Ob, At, I)$ and a set of attributes $B \subseteq At$

Output Decide whether B is a good attribute subset of $\mathcal{K}$

It is still possible to effect improvements in deciding if a given set of attributes is a good attribute subset of a given context

- Distel, F., Sertkaya, B.: *On the complexity of enumerating pseudo-intents.* Discrete Applied Mathematics **159** (2011) 450–466.
- Kuznetsov, S., Obiedkov, S.: *Counting pseudo-intents and ♯P-completeness.* In Missaoui, R., Schmid, J. (Editors): *ICFCA 2006.* Springer-Verlag (2006) 306–308.
- Sertkaya, B.: *Some computational problems related to pseudo-intents.* In Ferré, S., Rudolph, S. (Editors): *ICFCA 2009.* Springer-Verlag (2009) 130–145.

It is still possible to effect improvements in enumerating the set of all good attribute subsets of a given context

- Distel, F., Sertkaya, B.: *On the complexity of enumerating pseudo-intents.* Discrete Applied Mathematics **159** (2011) 450–466.
- Kuznetsov, S., Obiedkov, S.: *Counting pseudo-intents and ♯P-completeness.* In Missaoui, R., Schmid, J. (Editors): *ICFCA 2006.* Springer-Verlag (2006) 306–308.
- Sertkaya, B.: *Some computational problems related to pseudo-intents.* In Ferré, S., Rudolph, S. (Editors): *ICFCA 2009.* Springer-Verlag (2009) 130–145.

Concept algebras

Example 41:

	small	*medium*	*large*	*near*	*far*	*yes*	*no*
Mercury	⊗			⊗			⊗
Venus	⊗			⊗			⊗
Earth	⊗			⊗		⊗	
Mars	⊗			⊗		⊗	
Jupiter			⊗		⊗	⊗	
Saturn			⊗		⊗	⊗	
Uranus		⊗			⊗	⊗	
Neptune		⊗			⊗	⊗	
Pluto	⊗				⊗	⊗	

Formal context: structure of the forme $\mathcal{K} = (Ob, At, I)$ where

- *Ob* is a nonempty set of formal objects
- *At* is a nonempty set of formal attributes
- *I* is a binary relation between *Ob* and *At*

Example 41: Within the context of the planets

- $Ob = \{Mercury, Venus, \ldots\}$
- $At = \{small, medium, \ldots\}$
- $I = \{(Mercury, small), (Mercury, near), \ldots\}$

Concept algebras

Join and meet of concepts

For a set $A \subseteq Ob$ of objects, we define

- $A' = \{x \in At\colon X\ I\ x \text{ for every } X \in A\}$

i.e. the set of attributes common to the objects in A

For a set $B \subseteq At$ of attributes, we define

- $B' = \{X \in Ob\colon X\ I\ x \text{ for every } x \in B\}$

i.e. the set of objects which have all attributes in B

Example 41: Within the context of the planets

- $\{Earth, Mars\}' = \{small, near, yes\}$
- $\{small, near\}' = \{Mercury, Venus, Earth, Mars\}$

Concept algebras

Join and meet of concepts

A formal concept of the context (Ob, At, I) is a pair (A, B) with

- $A \subseteq Ob$
- $B \subseteq At$
- $A' = B$
- $B' = A$

Example 41: Within the context of the planets

- $(\{Earth, Mars\}, \{small, near, yes\})$
- $(\{Mercury, Venus, Earth, Mars\}, \{small, near\})$

Concept algebras

Join and meet of concepts

If (A_1, B_1) and (A_2, B_2) are concepts of a context then

- $A_1 \subseteq A_2$ iff $B_2 \subseteq B_1$

If $A_1 \subseteq A_2$ and $B_2 \subseteq B_1$ then we say that

- (A_1, B_1) is a subconcept of (A_2, B_2)
- (A_2, B_2) is a superconcept of (A_1, B_1)

and we write

- $(A_1, B_1) \leqslant (A_2, B_2)$

The set of all concepts of (Ob, At, I) ordered in this way

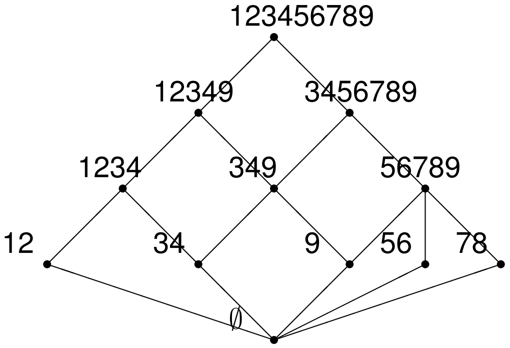
- is denoted by $\underline{\mathcal{B}}(Ob, At, I)$
- is called the concept lattice of the context (Ob, At, I)

Example 41:

	small	*medium*	*large*	*near*	*far*	*yes*	*no*
Mercury	$\otimes$			$\otimes$			$\otimes$
Venus	$\otimes$			$\otimes$			$\otimes$
Earth	$\otimes$			$\otimes$		$\otimes$	
Mars	$\otimes$			$\otimes$		$\otimes$	
Jupiter			$\otimes$		$\otimes$	$\otimes$	
Saturn			$\otimes$		$\otimes$	$\otimes$	
Uranus		$\otimes$			$\otimes$	$\otimes$	
Neptune		$\otimes$			$\otimes$	$\otimes$	
Pluto	$\otimes$				$\otimes$	$\otimes$	

Mercury = 1, *Venus* = 2, *Earth* = 3, *Mars* = 4, *Jupiter* = 5, *Saturn* = 6, *Uranus* = 7, *Neptune* = 8 et *Pluto* = 9

Example 41: Within the context of the planets (*Mercury* = 1, *Venus* = 2, *Earth* = 3, *Mars* = 4, *Jupiter* = 5, *Saturn* = 6, *Uranus* = 7, *Neptune* = 8 et *Pluto* = 9)



Concept algebras
Join and meet of concepts

Example 42:

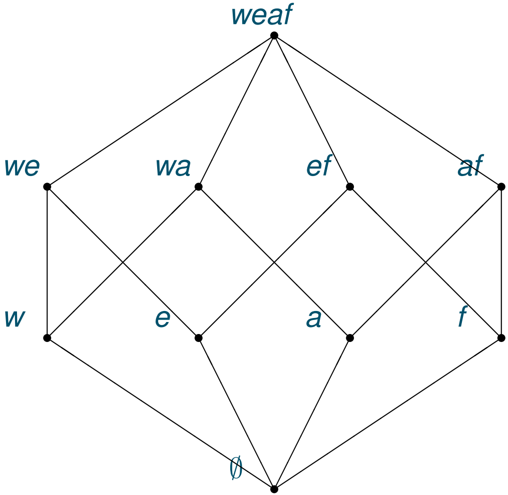
	cold	*moist*	*dry*	*warm*
water	$\times$	$\times$		
earth	$\times$		$\times$	
air		$\times$		$\times$
fire			$\times$	$\times$

$water = w$, $earth = e$, $air = a$ et $fire = f$

Concept algebras

Join and meet of concepts

Example 42: *water* = *w*, *earth* = *e*, *air* = *a* et *fire* = *f*



Theorem 5: The concept lattice $\underline{\mathcal{B}}(Ob, At, I)$ is a complete lattice in which infimum and supremum are given by

- $\bigwedge_{t \in T}(A_t, B_t) = (\bigcap_{t \in T} A_t, (\bigcup_{t \in T} B_t)'')$
- $\bigvee_{t \in T}(A_t, B_t) = ((\bigcup_{t \in T} A_t)'', \bigcap_{t \in T} B_t)$

Example: the concept “piano”

extent : the piano of Ray Charles, the piano of Diana Krall, etc

intent : to have a keyboard, to have pedals, etc

What is the negation of the concept “piano” ?

extent : the objects that do not possess one of the attributes of the concept “piano” ?

intent : the attributes that are not possessed by one of the objects of the concept “piano” ?

The negation of the concept
$(\{Earth, Mars\}, \{small, near, yes\})$:
$(\{Mercury, Venus, Jupiter, Saturn, Uranus, Neptune, Pluto\}, ?)$

	small	*medium*	*large*	*near*	*far*	*yes*	*no*
Mercury	×			×			×
Venus	×			×			×
Earth	×			×		×	
Mars	×			×		×	
Jupiter			×		×	×	
Saturn			×		×	×	
Uranus		×			×	×	
Neptune		×			×	×	
Pluto	×				×	×	

The negation of the concept
$(\{Earth, Mars\}, \{small, near, yes\})$:
$(?, \{medium, large, far, no\})$

	small	*medium*	*large*	*near*	*far*	*yes*	*no*
Mercury	×			×			×
Venus	×			×			×
Earth	×			×		×	
Mars	×			×		×	
Jupiter			×		×	×	
Saturn			×		×	×	
Uranus		×			×	×	
Neptune		×			×	×	
Pluto	×				×	×	

Concept algebras

Join, meet and complement of concepts

The negation of the concept
$(\{Mercury, Venus, Earth, Mars\}, \{small, near\})$:
$(\{Jupiter, Saturn, Uranus, Neptune, Pluto\}, ?)$

	small	*medium*	*large*	*near*	*far*	*yes*	*no*
Mercury	$\times$			$\times$			$\times$
Venus	$\times$			$\times$			$\times$
Earth	$\times$			$\times$		$\times$	
Mars	$\times$			$\times$		$\times$	
<u>Jupiter</u>			$\times$		$\otimes$	$\otimes$	
<u>Saturn</u>			$\times$		$\otimes$	$\otimes$	
<u>Uranus</u>		$\times$			$\otimes$	$\otimes$	
<u>Neptune</u>		$\times$			$\otimes$	$\otimes$	
<u>Pluto</u>	$\times$				$\otimes$	$\otimes$	

The negation of the concept
$(\{Mercury, Venus, Earth, Mars\}, \{small, near\})$:
$(?, \{medium, large, far, yes, no\})$

	small	*medium*	*large*	*near*	*far*	*yes*	*no*
Mercury	×			×			×
Venus	×			×			×
Earth	×			×		×	
Mars	×			×		×	
Jupiter			×		×	×	
Saturn			×		×	×	
Uranus		×			×	×	
Neptune		×			×	×	
Pluto	×				×	×	

Join of concepts (A_1, B_1) and (A_2, B_2)

- $((A_1 \cup A_2)'', B_1 \cap B_2)$

Meet of concepts (A_1, B_1) and (A_2, B_2)

- $(A_1 \cap A_2, (B_1 \cup B_2)'')$

Complement of concept (A, B)

- $(Obj \setminus A, -)$? No since • is not always an extent
- $(-, Att \setminus B)$? No since • is not always an intent
- $((Obj \setminus A)'', (Obj \setminus A)')$? No since • may intersect A
- $((Att \setminus B)', (Att \setminus B)'')$? No since • may intersect B

Concept algebras

Join, meet and complement of concepts

Example 42:

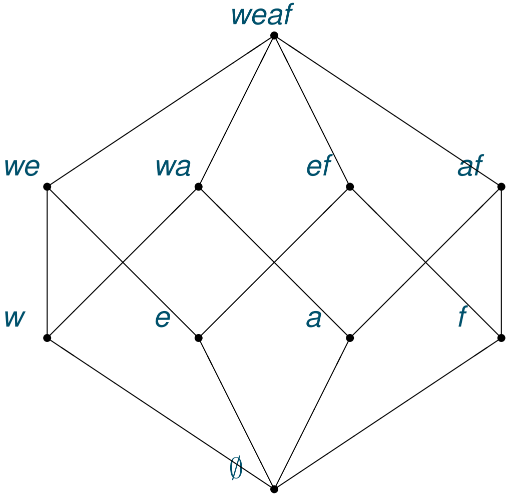
	cold	*moist*	*dry*	*warm*
water	$\times$	$\times$		
earth	$\times$		$\times$	
air		$\times$		$\times$
fire			$\times$	$\times$

$water = w$, $earth = e$, $air = a$ et $fire = f$

Concept algebras

Join, meet and complement of concepts

Example 42: *water* $=$ *w*, *earth* $=$ *e*, *air* $=$ *a* et *fire* $=$ *f*



Contexts

- $\mathcal{K} = (Ob, At, I)$ be a context
- $A \subseteq Ob$ be a set of objects
- $B \subseteq At$ be a set of attributes

Concepts

- (A, B) is a $\mathcal{H}$-concept iff $B' = A$ and $A' = B$

Semiconcepts

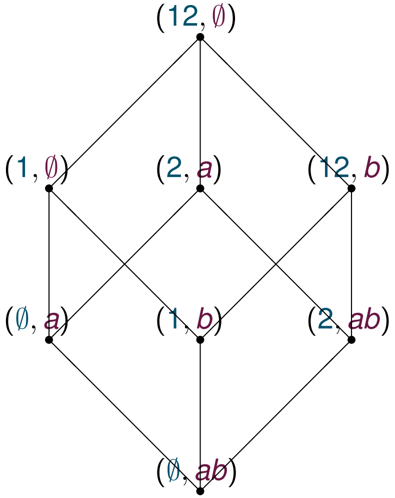
- (A, B) is a $\mathcal{H}$-semiconcept iff $B' = A$ or $A' = B$

Protoconcepts

- (A, B) is a $\mathcal{H}$-protoconcept iff $B' = A''$ or $A' = B''$

Example 43:

	a	b
1		×
2	×	×



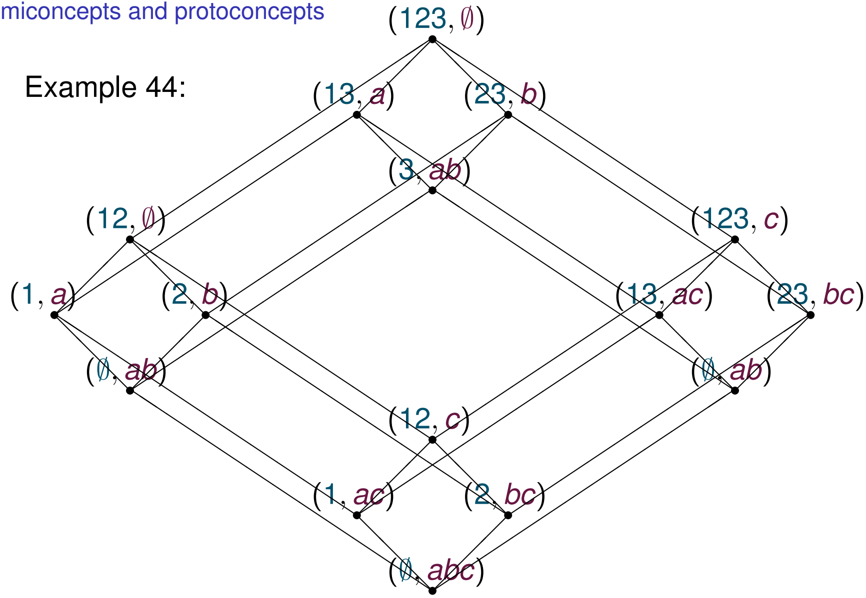
Example 44:

	a	b	c
1	$\times$		$\times$
2		$\times$	$\times$
3	$\times$	$\times$	$\times$

Concept algebras

Semiconcepts and protoconcepts

Example 44:

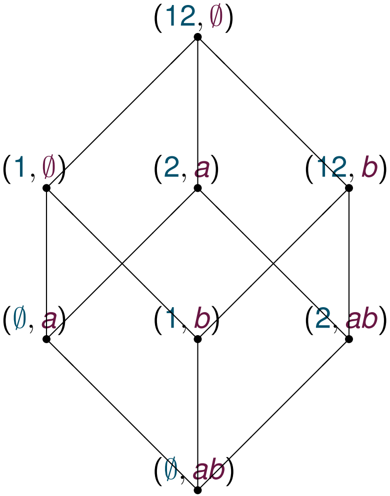


Structure
$\mathcal{A}(\mathcal{H}) = (A^{\mathcal{H}}, \bot_l^{\mathcal{H}}, \top_r^{\mathcal{H}}, \top_l^{\mathcal{H}}, \bot_r^{\mathcal{H}}, \neg_l^{\mathcal{H}}, \neg_r^{\mathcal{H}}, \vee_l^{\mathcal{H}}, \wedge_r^{\mathcal{H}}, \wedge_l^{\mathcal{H}}, \vee_r^{\mathcal{H}})$ where
$A^{\mathcal{H}}$ is the set of all $\mathcal{H}$'s protoconcepts and

- $\bot_l^{\mathcal{H}} = (\emptyset, At)$
- $\top_r^{\mathcal{H}} = (Ob, \emptyset)$
- $\top_l^{\mathcal{H}} = (Ob, Ob')$
- $\bot_r^{\mathcal{H}} = (At', At)$
- $\neg_l^{\mathcal{H}}(A, B) = (Ob \setminus A, (Ob \setminus A)')$
- $\neg_r^{\mathcal{H}}(A, B) = ((At \setminus B)', At \setminus B)$
- $(A_1, B_1) \vee_l^{\mathcal{H}} (A_2, B_2) = (A_1 \cup A_2, (A_1 \cup A_2)')$
- $(A_1, B_1) \wedge_r^{\mathcal{H}} (A_2, B_2) = ((B_1 \cup B_2)', B_1 \cup B_2)$
- $(A_1, B_1) \wedge_l^{\mathcal{H}} (A_2, B_2) = (A_1 \cap A_2, (A_1 \cap A_2)')$
- $(A_1, B_1) \vee_r^{\mathcal{H}} (A_2, B_2) = ((B_1 \cap B_2)', B_1 \cap B_2)$

Example 45:

	a	b
1		$\times$
2	$\times$	$\times$



Concept algebras

Protoconcept algebras

- $\wedge_l$ is AC — $\vee_r$ is AC
- $\wedge_l$ distributes over $\vee_l$ — $\vee_r$ distributes over $\wedge_r$
- $\neg_l(x \wedge_l x) = \neg_l x$ — $\neg_r(x \vee_r x) = \neg_r x$
- $x \wedge_l (y \wedge_l y) = x \wedge_l y$ — $x \vee_r (y \vee_r y) = x \vee_r y$
- $x \wedge_l (x \vee_l y) = x \wedge_l x$ — $x \vee_r (x \wedge_r y) = x \vee_r x$
- $x \wedge_l (x \vee_r y) = x \wedge_l x$ — $x \vee_r (x \wedge_l y) = x \vee_r x$
- $\neg_l(\neg_l x \wedge_l \neg_l y) = x \vee_l y$ — $\neg_r(\neg_r x \vee_r \neg_r y) = x \wedge_r y$
- $\neg_l \bot_l = \top_l$ — $\neg_r \top_r = \bot_r$
- $\neg_l \top_r = \bot_l$ — $\neg_r \bot_l = \top_r$
- $\top_r \wedge_l \top_r = \top_l$ — $\bot_l \vee_r \bot_l = \bot_r$
- $x \wedge_l \neg_l x = \bot_l$ — $x \vee_r \neg_r x = \top_r$
- $\neg_l \neg_l (x \wedge_l y) = x \wedge_l y$ — $\neg_r \neg_r (x \vee_r y) = x \vee_r y$
- $(x \vee_r x) \wedge_l (x \vee_r x) = (x \wedge_l x) \vee_r (x \wedge_l x)$

Let $\mathcal{H} = (Ob, At, I)$ be a **context**

- If $A^{\mathcal{H}}$ is the set of all $\mathcal{H}$'s protoconcepts then the structure $\mathcal{A}(\mathcal{H}) = (A^{\mathcal{H}}, \bot_l^{\mathcal{H}}, \top_r^{\mathcal{H}}, \neg_l^{\mathcal{H}}, \neg_r^{\mathcal{H}}, \vee_l^{\mathcal{H}}, \wedge_r^{\mathcal{H}})$ is a **protoconcept algebra**

Let $\mathcal{A} = (A, \bot_l, \top_r, \neg_l, \neg_r, \vee_l, \wedge_r)$ be a **protoconcept algebra**

- There exists a **context** $\mathcal{H}(\mathcal{A}) = (Ob^{\mathcal{A}}, At^{\mathcal{A}}, I^{\mathcal{A}})$ such that $\mathcal{A}$ is embeddable into the structure $\mathcal{A}(\mathcal{H}(\mathcal{A})) = (A^{\mathcal{H}(\mathcal{A})}, \bot_l^{\mathcal{H}(\mathcal{A})}, \top_r^{\mathcal{H}(\mathcal{A})}, \neg_l^{\mathcal{H}(\mathcal{A})}, \neg_r^{\mathcal{H}(\mathcal{A})}, \vee_l^{\mathcal{H}(\mathcal{A})}, \wedge_r^{\mathcal{H}(\mathcal{A})})$

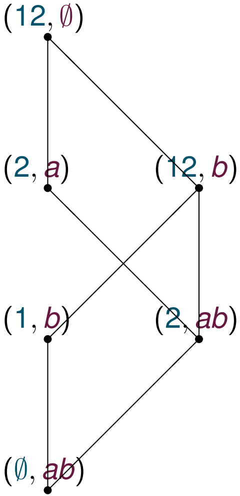
Structure

$\mathcal{A}(\mathcal{H}) = (A^{\mathcal{H}}, \bot_l^{\mathcal{H}}, \top_r^{\mathcal{H}}, \top_l^{\mathcal{H}}, \bot_r^{\mathcal{H}}, \neg_l^{\mathcal{H}}, \neg_r^{\mathcal{H}}, \vee_l^{\mathcal{H}}, \wedge_r^{\mathcal{H}}, \wedge_l^{\mathcal{H}}, \vee_r^{\mathcal{H}})$ where $A^{\mathcal{H}}$ is the set of all $\mathcal{H}$'s semiconcepts and

- $\bot_l^{\mathcal{H}} = (\emptyset, At)$
- $\top_r^{\mathcal{H}} = (Ob, \emptyset)$
- $\top_l^{\mathcal{H}} = (Ob, Ob')$
- $\bot_r^{\mathcal{H}} = (At', At)$
- $\neg_l^{\mathcal{H}}(A, B) = (Ob \setminus A, (Ob \setminus A)')$
- $\neg_r^{\mathcal{H}}(A, B) = ((At \setminus B)', At \setminus B)$
- $(A_1, B_1) \vee_l^{\mathcal{H}} (A_2, B_2) = (A_1 \cup A_2, (A_1 \cup A_2)')$
- $(A_1, B_1) \wedge_r^{\mathcal{H}} (A_2, B_2) = ((B_1 \cup B_2)', B_1 \cup B_2)$
- $(A_1, B_1) \wedge_l^{\mathcal{H}} (A_2, B_2) = (A_1 \cap A_2, (A_1 \cap A_2)')$
- $(A_1, B_1) \vee_r^{\mathcal{H}} (A_2, B_2) = ((B_1 \cap B_2)', B_1 \cap B_2)$

Example 46:

	a	b
1		$\times$
2	$\times$	$\times$



Concept algebras

Semiconcept algebras

- $\wedge_l$ is AC — $\vee_r$ is AC
- $\wedge_l$ distributes over $\vee_l$ — $\vee_r$ distributes over $\wedge_r$
- $\neg_l(x \wedge_l x) = \neg_l x$ — $\neg_r(x \vee_r x) = \neg_r x$
- $x \wedge_l (y \wedge_l y) = x \wedge_l y$ — $x \vee_r (y \vee_r y) = x \vee_r y$
- $x \wedge_l (x \vee_l y) = x \wedge_l x$ — $x \vee_r (x \wedge_r y) = x \vee_r x$
- $x \wedge_l (x \vee_r y) = x \wedge_l x$ — $x \vee_r (x \wedge_l y) = x \vee_r x$
- $\neg_l(\neg_l x \wedge_l \neg_l y) = x \vee_l y$ — $\neg_r(\neg_r x \vee_r \neg_r y) = x \wedge_r y$
- $\neg_l \bot_l = \top_l$ — $\neg_r \top_r = \bot_r$
- $\neg_l \top_r = \bot_l$ — $\neg_r \bot_l = \top_r$
- $\top_r \wedge_l \top_r = \top_l$ — $\bot_l \vee_r \bot_l = \bot_r$
- $x \wedge_l \neg_l x = \bot_l$ — $x \vee_r \neg_r x = \top_r$
- $\neg_l \neg_l (x \wedge_l y) = x \wedge_l y$ — $\neg_r \neg_r (x \vee_r y) = x \vee_r y$
- $$(x \vee_r x) \wedge_l (x \vee_r x) = (x \wedge_l x) \vee_r (x \wedge_l x)$$
- $$x \wedge_l x = x \text{ or } x \vee_r x = x$$

Let $\mathcal{H} = (Ob, At, I)$ be a **context**

- If $A^{\mathcal{H}}$ is the set of all $\mathcal{H}$'s semiconcepts then the structure $\mathcal{A}(\mathcal{H}) = (A^{\mathcal{H}}, \bot_l^{\mathcal{H}}, \top_r^{\mathcal{H}}, \neg_l^{\mathcal{H}}, \neg_r^{\mathcal{H}}, \vee_l^{\mathcal{H}}, \wedge_r^{\mathcal{H}})$ is a **semiconcept algebra**

Let $\mathcal{A} = (A, \bot_l, \top_r, \neg_l, \neg_r, \vee_l, \wedge_r)$ be a **semiconcept algebra**

- There exists a **context** $\mathcal{H}(\mathcal{A}) = (Ob^{\mathcal{A}}, At^{\mathcal{A}}, I^{\mathcal{A}})$ such that $\mathcal{A}$ is embeddable into the structure $\mathcal{A}(\mathcal{H}(\mathcal{A})) = (A^{\mathcal{H}(\mathcal{A})}, \bot_l^{\mathcal{H}(\mathcal{A})}, \top_r^{\mathcal{H}(\mathcal{A})}, \neg_l^{\mathcal{H}(\mathcal{A})}, \neg_r^{\mathcal{H}(\mathcal{A})}, \vee_l^{\mathcal{H}(\mathcal{A})}, \wedge_r^{\mathcal{H}(\mathcal{A})})$

Concept algebras

The word problem

We define terms as follows

- $s ::= x \mid 0_l \mid 1_r \mid -_l s \mid -_r s \mid (s \sqcup_l t) \mid (s \sqcap_r t)$

We define the following abbreviations

- $1_l ::= -_l 0_l$
- $0_r ::= -_r 1_r$
- $(s \sqcap_l t) ::= -_l(-_l s \sqcup_l -_l t)$
- $(s \sqcup_r t) ::= -_r(-_r s \sqcap_r -_r t)$

Concept algebras

The word problem

A valuation based on a protoconcept algebra / semiconcept algebra $\mathcal{A} = (A, \bot_l, \top_r, \neg_l, \neg_r, \vee_l, \wedge_r)$ is a function

- $\theta \colon x \mapsto \theta(x) \in A$

θ induces a function $\bar{\theta} \colon s \mapsto \bar{\theta}(s) \in A$ as follows:

- $\bar{\theta}(x) = \theta(x)$
- $\bar{\theta}(0_l) = \bot_l$
- $\bar{\theta}(1_r) = \top_r$
- $\bar{\theta}(-_l s) = \neg_l \bar{\theta}(s)$
- $\bar{\theta}(-_r s) = \neg_r \bar{\theta}(s)$
- $\bar{\theta}(s \sqcup_l t) = \bar{\theta}(s) \vee_l \bar{\theta}(t)$
- $\bar{\theta}(s \sqcap_r t) = \bar{\theta}(s) \wedge_r \bar{\theta}(t)$

We consider the following problem: Deciding whether two terms are equivalent in every protoconcept algebras / semiconcept algebras

Input Terms s, t

Output Decide whether $s \not\approx t$, i.e. whether there exists a valuation θ based on a protoconcept algebra / semiconcept algebra $\mathcal{A} = (A, \bot_l, \top_r, \neg_l, \neg_r, \vee_l, \wedge_r)$ such that $\bar{\theta}(s) \neq \bar{\theta}(t)$

The exact computational complexity of the above problem is unknown

- Herrmann, C., Luksch, P., Skorsky, M., Wille, R.: *Algebras of semiconcepts and double Boolean algebras.* Technische Universität Darmstadt (2000).
- Vormbrock, B.: *A solution of the word problem for free double Boolean algebras.* In Kuznetsov, S., Schmidt, S. (Editors): *ICFCA 2007.* Springer-Verlag (2007) 240–270.

The exact computational complexity of the above problem is unknown

- Vormbrock, B., Wille, R.: *Semiconcept and protoconcept algebras: the basic theorems.* In Ganter, B., Stumme, G., Wille, R. (Editors): *Formal Concept Analysis.* Springer-Verlag (2005) 34–48.
- Wille, R.: *Boolean concept logic.* In Eklund, P. (Editor): *ICFCA 2004.* Springer-Verlag (2004) 1–13.

Concepts and roles

Description logics: syntax

Knowledge base terminological box (TBox) $+$ assertional box (ABox)

TBox

- terminology of an application domain
- set of concept definitions of the form $A \equiv C$ and general concept inclusion of the form $C \sqsubseteq D$
- $A \equiv C$ assigns the concept name A to the concept description C
- $C \sqsubseteq D$ states a subconcept/superconcept relationship between C and D

Example 47: Example of a TBox

- $\mathcal{T} := \{$
  $\mathit{LandlockedCountry} \equiv \mathit{Country} \sqcap \forall \mathit{hasBorderTo}.\mathit{Land},$
  $\mathit{OceanCountry} \equiv \mathit{Country} \sqcap \exists \mathit{hasBorderTo}.\mathit{Ocean}$
  $\}$

Description logics: syntax

Knowledge base terminological box (TBox) $+$ assertional box (ABox)

ABox

- facts about a specific world
- set of concept assertions of the form $C(a)$ and role assertions of the form $R(a, b)$
- $C(a)$ means that the individual a is an instance of the concept C
- $R(a, b)$ means that the individual a is in R-relation with individual b

Example 48: Example of an ABox

- $\mathcal{A} := \{$
  LandlockedCountry(*Austria*),
  Country(*Portugal*),
  Ocean(*Atlantic*),
  hasBorderTo(*Portugal*, *Atlantic*)
  $\}$

Description logics: semantics

Interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$

Domain $\Delta^{\mathcal{I}}$ is a nonempty set

Interpretation function $\cdot^{\mathcal{I}}$ maps

- every concept occurring in the TBox to a subset of the domain
- every individual name occurring in the ABox to an element of the domain
- every role to a binary relation on the domain

Example 49:

- $\mathcal{T} := \{$
  $LandlockedCountry \equiv Country \sqcap \forall hasBorderTo.Land,$
  $OceanCountry \equiv Country \sqcap \exists hasBorderTo.Ocean$
  $\}$
- $\mathcal{A} := \{$
  $LandlockedCountry(Austria),$
  $Country(Portugal),$
  $Ocean(Atlantic),$
  $hasBorderTo(Portugal, Atlantic)$
  $\}$

Description logics: inferences

- Given an explicit TBox and an explicit ABox, deduce implicit consequences such as

  Subsumption problem $C \sqsubseteq D$
  Instance checking $C(a)$

Concepts and roles

The basic description language *AL*

Let $A_1, A_2, \ldots$ be atomic concepts and $A_1, A_2, \ldots$ atomic roles

Concepts are formed by means of the rule

- $C ::= A \mid \top \mid \bot \mid \neg A \mid (C \sqcap D) \mid \forall R.C \mid \exists R.\top$

Interpretation

Interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$

Domain $\Delta^{\mathcal{I}}$ is a nonempty set

Interpretation function $\cdot^{\mathcal{I}}$ maps

- every atomic concept A to a subset $A^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$
- every atomic role R to a binary relation $R^{\mathcal{I}}$ on $\Delta^{\mathcal{I}}$
- $\top$ to $\Delta^{\mathcal{I}}$ and $\bot$ to $\emptyset$
- $\neg A$ to $\Delta^{\mathcal{I}} \setminus A^{\mathcal{I}}$ and $C \sqcap D$ to $C^{\mathcal{I}} \cap D^{\mathcal{I}}$
- $\forall R.C$ to $\{a \in \Delta^{\mathcal{I}}$: for all $b \in \Delta^{\mathcal{I}}$, if $R^{\mathcal{I}}(a, b)$ then $b \in C^{\mathcal{I}}\}$ and $\exists R.\top$ to $\{a \in \Delta^{\mathcal{I}}$: there exists $b \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(a, b)\}$

Concepts and roles

The basic description language $\mathcal{AL}$

We say that two concepts C and D are equivalent, in symbols $C \equiv D$, iff

- $C^{\mathcal{I}} = D^{\mathcal{I}}$ for all interpretations I

Example 50:

- $\forall hasChild.Female \sqcap \forall hasChild.Student \equiv \forall hasChild.(Female \sqcap Student)$

Concepts and roles

We obtain more expressive languages if we add further constructors

negation $\neg C$ interpreted in $\mathcal{I}$ by $\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$

union $(C \sqcup D)$ interpreted in $\mathcal{I}$ by $C^{\mathcal{I}} \cup D^{\mathcal{I}}$

full existential quantification $\exists R.C$ interpreted in $\mathcal{I}$ by $\{a \in \Delta^{\mathcal{I}}$: there exists $b \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(a, b)$ and $b \in C^{\mathcal{I}}\}$

at-least restriction $(\geqslant n\ R)$ interpreted in $\mathcal{I}$ by $\{a \in \Delta^{\mathcal{I}}$: there exists at least n $b \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(a, b)$ and $b \in C^{\mathcal{I}}\}$

at-most restriction $(\leqslant n\ R)$ interpreted in $\mathcal{I}$ by $\{a \in \Delta^{\mathcal{I}}$: there exists at most n $b \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(a, b)$ and $b \in C^{\mathcal{I}}\}$

Concepts and roles

The family of $\mathcal{AL}$-languages

We obtain more expressive languages if we add further constructors

role intersection $R \sqcap S$ interpreted in $\mathcal{I}$ by $R^{\mathcal{I}} \cap S^{\mathcal{I}}$

role composition $R; S$ interpreted in $\mathcal{I}$ by $R^{\mathcal{I}} \circ S^{\mathcal{I}}$

transitive closure of a role R^{+} interpreted in $\mathcal{I}$ by the transitiveclosure of $R^{\mathcal{I}}$

role inverse R^{-1} interpreted in $\mathcal{I}$ by the inverse of $R^{\mathcal{I}}$

Example:

- $\exists(\mathit{hasSon} \sqcup \mathit{hasDaughter})^{+\,-1}.(\mathit{Woman} \sqcap \mathit{Mathematician})$

Example 51:

- $Person \sqcap (\leqslant 1\ hasChild \sqcup (\geqslant 3\ hasChild \sqcap \exists hasChild.Female))$

Concepts and roles

Terminologies

Terminological axioms have the form

- $C \sqsubseteq D$, $C \equiv D$, $R \sqsubseteq S$, $R \equiv S$

Concept definitions have the form

- $A \equiv C$

Example 52:

- $Mother \equiv Woman \sqcap \exists hasChild.Person$
- $Parent \equiv Mother \sqcup Father$

Example 53: A terminology (TBox) with concepts about family relationships

- $Woman \equiv Person \sqcap Female$
- $Man \equiv Person \sqcap \neg Female$
- $Mother \equiv Woman \sqcap \exists hasChild.Person$
- $Father \equiv Man \sqcap \exists hasChild.Person$
- $Parent \equiv Mother \sqcup Father$
- $Grandmother \equiv Mother \sqcap \exists hasChild.Parent$
- $MotherWithManyChildren \equiv Mother \sqcap \geqslant 3 hasChild$
- $MotherWithoutDaughter \equiv Mother \sqcap \forall hasChild.\neg Woman$
- $Wife \equiv Woman \sqcap \exists hasHusband.Man$

Cyclic definitions in TBox

- $Human \equiv Animal \sqcap \forall hasParent.Human$
- $ManOnlyMaleDescendants \equiv$ $Man \sqcap \forall hasChild.ManOnlyMaleDescendants$
- $BinaryTree \equiv Tree \sqcap \leqslant$ $2\ hasBranch \sqcap \forall hasBranch.BinaryTree$

The second component of a knowledge base, in addition to the TBox, is the world description or ABox

- $C(a)$, $R(a, b)$

Example 54: A world descrption (ABox)

- $MotherWithoutDaughter(MARY)$
- $hasChild(MARY, PAUL)$
- $hasChild(MARY, PETER)$
- $Father(PETER)$
- $hasChild(PETER, HARRY)$

Concepts and roles

Inferences

The different kinds of reasoning performed by a DL systems are

- checking satisfiability of concepts: a concept C is satisfiable with respect to T if there exists a model $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ of T such that $C^{\mathcal{I}}$ is nonempty
- checking subsumption of concepts: a concept C is subsumed by a concept D with respect to T if for every model $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ of T, $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$
- checking equivalence of concepts: a concept C is equivalent to a concept D with respect to T if for every model $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ of T, $C^{\mathcal{I}} = D^{\mathcal{I}}$

- Baader, F., Calvanese, D., McGuinness, D., Nardi, D., Patel-Schneider, P. (Editors): *The Description Logic Handbook.* Cambridge University Press (2003).
- Sertkaya, B.: *A survey on how description logic ontologies benefit from FCA.* In Kryszkiewicz, M., Obiedkov, S. (Editors): Proceedings of the 7th International Conference on Concept Lattices and Their Applications (CLA 2010). Volume 672 of CEUR Workshop Proceedings (2010) 2–21.

Research problems

Generalize to ternary contexts the techniques in formal concept analysis that are presented in these slides

Generalize to fuzzy/probabilistic/possibilistic contexts the techniques in formal concept analysis that are presented in these slides

Effect improvements in finding all concepts of a given context

Effect improvements in drawing the concept lattice of a given context

Research problems

Effect improvements in deciding if a given set of attributes is a good attribute subset of a given context

Effect improvements in enumerating the set of all good attribute subsets of a given context

Exact computational complexity of deciding whether two terms are equivalent in every protoconcept algebras / semiconcept algebras

Enrich formal concept analysis with description logic constructors and apply formal concept analysis methods in description logics